

An introduction to d-manifolds and derived differential geometry

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Abstract

This is a survey of the author's book [21]. We introduce a 2-category \mathbf{dMan} of *d-manifolds*, new geometric objects which are 'derived' smooth manifolds, in the sense of the 'derived algebraic geometry' of Toën and Lurie. Manifolds \mathbf{Man} embed in \mathbf{dMan} as a full (2-)subcategory. There are also 2-categories $\mathbf{dMan}^b, \mathbf{dMan}^c$ of *d-manifolds with boundary* and *with corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb}^b, \mathbf{dOrb}^c$, *d-orbifolds*.

Much of differential geometry extends very nicely to d-manifolds — immersions, submersions, submanifolds, transverse fibre products, orientations, etc. Compact oriented d-manifolds have virtual classes.

Many areas of symplectic geometry involve 'counting' moduli spaces $\bar{\mathcal{M}}_{g,m}(J, \beta)$ of *J*-holomorphic curves to define invariants, Floer homology theories, etc. Such $\bar{\mathcal{M}}_{g,m}(J, \beta)$ are given the structure of *Kuranishi spaces* in the work of Fukaya, Oh, Ohta and Ono [10], but there are problems with the theory. The author believes the 'correct' definition of Kuranishi spaces is that they are d-orbifolds with corners. D-manifolds and d-orbifolds will have applications in symplectic geometry, and elsewhere.

For brevity, this survey focusses on d-manifolds without boundary. A longer and more detailed summary of the book is given in [22].

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1 Introduction

This is a survey of [21], and describes the author’s new theory of ‘derived differential geometry’. The objects in this theory are *d-manifolds*, ‘derived’ versions of smooth manifolds, which form a (strict) 2-category **dMan**. There are also 2-categories of *d-manifolds with boundary* **dMan^b** and *d-manifolds with corners* **dMan^c**, and orbifold versions of all these, *d-orbifolds* **dOrb**, **dOrb^b**, **dOrb^c**.

Here ‘derived’ is intended in the sense of *derived algebraic geometry*. The original motivating idea for derived algebraic geometry, as in Kontsevich [23] for instance, was that certain moduli schemes \mathcal{M} appearing in enumerative invariant problems may be highly singular as schemes. However, it may be natural to realize \mathcal{M} as a truncation of some ‘derived’ moduli space \mathcal{M} , a new kind of geometric object living in a higher category. The geometric structure on \mathcal{M} should encode the full deformation theory of the moduli problem, the obstructions as well as the deformations. It was hoped that \mathcal{M} would be ‘smooth’, and so in some sense simpler than its truncation \mathcal{M} .

Early work in derived algebraic geometry focussed on *dg-schemes*, as in Ciocan-Fontanine and Kapranov [7]. These have largely been replaced by the *derived stacks* of Toën and Vezzosi [33,34], and the *structured spaces* of Lurie [25, 26]. *Derived differential geometry* aims to generalize these ideas to differential geometry and smooth manifolds. A brief note about it can be found in Lurie [26, §4.5]; the ideas are worked out in detail by Lurie’s student David Spivak [31], who defines an ∞ -category (simplicial category) of *derived manifolds*.

The author came to these questions from a different direction, symplectic geometry. Many important areas in symplectic geometry involve forming moduli spaces $\overline{\mathcal{M}}_{g,m}(J, \beta)$ of *J*-holomorphic curves in some symplectic manifold (M, ω) , possibly with boundary in a Lagrangian *L*, and then ‘counting’ these moduli spaces to get ‘invariants’ with interesting properties. Such areas include

Gromov–Witten invariants (open and closed), Lagrangian Floer cohomology, Symplectic Field Theory, contact homology, and Fukaya categories.

To do this ‘counting’, one needs to put a suitable geometric structure on $\bar{\mathcal{M}}_{g,m}(J, \beta)$ — something like the ‘derived’ moduli spaces \mathcal{M} above — and use this to define a ‘virtual class’ or ‘virtual chain’ in \mathbb{Z}, \mathbb{Q} or some homology theory. There is no general agreement on what geometric structure to use — compared to the elegance of algebraic geometry, this area is something of a mess. Two rival theories for geometric structures to put on moduli spaces $\bar{\mathcal{M}}_{g,m}(J, \beta)$ are the *Kuranishi spaces* of Fukaya, Oh, Ohta and Ono [10, 11] and the *polyfolds* of Hofer, Wysocki and Zehnder [13–15].

The theory of Kuranishi spaces in [10, 11] does not go far — they define Kuranishi spaces, and construct virtual cycles upon them, but they do not define morphisms between Kuranishi spaces, for instance. The author tried to study Kuranishi spaces as geometric spaces in their own right, but ran into problems, and became convinced that a new definition of Kuranishi space was needed. Upon reading Spivak’s theory of derived manifolds [31], it became clear that some form of ‘derived differential geometry’ was required: *Kuranishi spaces in the sense of [10, §A] ought to be defined to be ‘derived orbifolds with corners’*.

The author tried to read Lurie [25, 26] and Spivak [31] with a view to applications to Kuranishi spaces and symplectic geometry, but ran into problems of a different kind: the framework of [25, 26, 31] is formidably long, complex and abstract, and proved too difficult for a humble trainee symplectic geometer to understand, or use as a tool. So the author looked for a way to simplify the theory, while retaining the information needed for applications to symplectic geometry. The theory of d-manifolds and d-orbifolds of [21] is the result.

The essence of our simplification is this. Consider a ‘derived’ moduli space \mathcal{M} of some objects E , e.g. vector bundles on some \mathbb{C} -scheme X . One expects \mathcal{M} to have a ‘cotangent complex’ $\mathbb{L}_{\mathcal{M}}$, a complex in some derived category with cohomology $h^i(\mathbb{L}_{\mathcal{M}})|_E \cong \text{Ext}^{1-i}(E, E)^*$ for $i \in \mathbb{Z}$. In general, $\mathbb{L}_{\mathcal{M}}$ can have nontrivial cohomology in many negative degrees, and because of this such objects \mathcal{M} must form an ∞ -category to properly describe their geometry.

However, the moduli spaces relevant to enumerative invariant problems are of a restricted kind: one considers only \mathcal{M} such that $\mathbb{L}_{\mathcal{M}}$ has nontrivial cohomology only in degrees $-1, 0$, where $h^0(\mathbb{L}_{\mathcal{M}})$ encodes the (dual of the) deformations $\text{Ext}^1(E, E)^*$, and $h^{-1}(\mathbb{L}_{\mathcal{M}})$ the (dual of the) obstructions $\text{Ext}^2(E, E)^*$. As in Toën [33, §4.4.3], such derived spaces are called *quasi-smooth*, and this is a necessary condition on \mathcal{M} for the construction of a virtual fundamental class.

Our construction of d-manifolds replaces complexes in a derived category $D^b \text{coh}(\mathcal{M})$ with a 2-category of complexes in degrees $-1, 0$ only. For general \mathcal{M} this loses a lot of information, but for quasi-smooth \mathcal{M} , since $\mathbb{L}_{\mathcal{M}}$ is concentrated in degrees $-1, 0$, the important information is retained. In the language of Toën and Vezzosi [33, 34], this corresponds to working with a subclass of derived schemes whose dg-algebras are of a special kind: they are 2-step supercommutative dg-algebras $A^{-1} \xrightarrow{d} A^0$ such that $d(A^{-1}) \cdot A^{-1} = 0$. Then $d(A^{-1})$ is a square zero ideal in A^0 , and A^{-1} is a module over $H^0(A^{-1} \xrightarrow{d} A^0)$.

The set up of [21] is also long and complicated. But mostly this complexity comes from other sources: working over C^∞ -rings, and including manifolds with boundary, manifolds with corners, and orbifolds. The 2-category style ‘derived geometry’ of [21] really is far simpler than those of [25, 26, 31, 33, 34].

Following Spivak [31], in order to be able to use the tools of algebraic geometry — schemes, stacks, quasicoherent sheaves — in differential geometry, our d-manifolds are built on the notions of C^∞ -ring and C^∞ -scheme that were invented in synthetic differential geometry, and developed further by the author in [19, 20]. We survey the C^∞ -algebraic geometry we need in §2. Section 3 discusses the 2-category of d -spaces **dSpa**, which are ‘derived’ C^∞ -schemes, and §4 describes the 2-category of d-manifolds **dMan**, and our theory of ‘derived differential geometry’. Appendix A explains the basics of 2-categories.

For brevity, and to get the main ideas across as simply as possible, this survey will concentrate on d-manifolds without boundary, apart from a short section on d-manifolds with corners and d-orbifolds in §4.9. A longer summary of the book, including much more detail on d-manifolds with corners, d-orbifolds, and d-orbifolds with corners, is given in [22].

Acknowledgements. I would like to thank Jacob Lurie for helpful conversations.

2 C^∞ -rings and C^∞ -schemes

If X is a manifold then the \mathbb{R} -algebra $C^\infty(X)$ of smooth functions $c : X \rightarrow \mathbb{R}$ is a C^∞ -ring. That is, for each smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ there is an n -fold operation $\Phi_f : C^\infty(X)^n \rightarrow C^\infty(X)$ acting by $\Phi_f : c_1, \dots, c_n \mapsto f(c_1, \dots, c_n)$, and these operations Φ_f satisfy many natural identities. Thus, $C^\infty(X)$ actually has a far richer algebraic structure than the obvious \mathbb{R} -algebra structure.

In [19] (surveyed in [20]) the author set out a version of algebraic geometry in which rings or algebras are replaced by C^∞ -rings, focussing on C^∞ -schemes, a category of geometric objects which generalize manifolds, and whose morphisms generalize smooth maps, *quasicoherent* and *coherent sheaves* on C^∞ -schemes, and C^∞ -stacks, in particular *Deligne–Mumford C^∞ -stacks*, a 2-category of geometric objects which generalize orbifolds. Much of the material on C^∞ -schemes was already known in synthetic differential geometry, see for instance Dubuc [9] and Moerdijk and Reyes [29].

2.1 C^∞ -rings

Definition 2.1. A C^∞ -ring is a set \mathfrak{C} together with operations $\Phi_f : \mathfrak{C}^n \rightarrow \mathfrak{C}$ for all $n \geq 0$ and smooth maps $f : \mathbb{R}^n \rightarrow \mathbb{R}$, where by convention when $n = 0$ we define \mathfrak{C}^0 to be the single point $\{\emptyset\}$. These operations must satisfy the following relations: suppose $m, n \geq 0$, and $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, m$ and $g : \mathbb{R}^m \rightarrow \mathbb{R}$ are smooth functions. Define a smooth function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$h(x_1, \dots, x_n) = g(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)),$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. Then for all $(c_1, \dots, c_n) \in \mathfrak{C}^n$ we have

$$\Phi_h(c_1, \dots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \dots, c_n), \dots, \Phi_{f_m}(c_1, \dots, c_n)).$$

We also require that for all $1 \leq j \leq n$, defining $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\pi_j : (x_1, \dots, x_n) \mapsto x_j$, we have $\Phi_{\pi_j}(c_1, \dots, c_n) = c_j$ for all $(c_1, \dots, c_n) \in \mathfrak{C}^n$.

Usually we refer to \mathfrak{C} as the C^∞ -ring, leaving the operations Φ_f implicit.

A *morphism* between C^∞ -rings $(\mathfrak{C}, (\Phi_f)_{f: \mathbb{R}^n \rightarrow \mathbb{R}})$, $(\mathfrak{D}, (\Psi_f)_{f: \mathbb{R}^n \rightarrow \mathbb{R}})$ is a map $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ such that $\Psi_f(\phi(c_1), \dots, \phi(c_n)) = \phi \circ \Phi_f(c_1, \dots, c_n)$ for all smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $c_1, \dots, c_n \in \mathfrak{C}$. We will write $\mathbf{C}^\infty\mathbf{Rings}$ for the category of C^∞ -rings.

Here is the motivating example:

Example 2.2. Let X be a manifold. Write $C^\infty(X)$ for the set of smooth functions $c : X \rightarrow \mathbb{R}$. For $n \geq 0$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth, define $\Phi_f : C^\infty(X)^n \rightarrow C^\infty(X)$ by

$$(\Phi_f(c_1, \dots, c_n))(x) = f(c_1(x), \dots, c_n(x)), \quad (2.1)$$

for all $c_1, \dots, c_n \in C^\infty(X)$ and $x \in X$. It is easy to see that $C^\infty(X)$ and the operations Φ_f form a C^∞ -ring.

Now let $f : X \rightarrow Y$ be a smooth map of manifolds. Then pullback $f^* : C^\infty(Y) \rightarrow C^\infty(X)$ mapping $f^* : c \mapsto c \circ f$ is a morphism of C^∞ -rings. Furthermore (at least for Y without boundary), every C^∞ -ring morphism $\phi : C^\infty(Y) \rightarrow C^\infty(X)$ is of the form $\phi = f^*$ for a unique smooth map $f : X \rightarrow Y$.

Write $\mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ for the opposite category of $\mathbf{C}^\infty\mathbf{Rings}$, with directions of morphisms reversed, and \mathbf{Man} for the category of manifolds without boundary. Then we have a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}} : \mathbf{Man} \rightarrow \mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$ acting by $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}}(X) = C^\infty(X)$ on objects and $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}}(f) = f^*$ on morphisms. This embeds \mathbf{Man} as a full subcategory of $\mathbf{C}^\infty\mathbf{Rings}^{\text{op}}$.

Note that C^∞ -rings are far more general than those coming from manifolds. For example, if X is any topological space we could define a C^∞ -ring $C^0(X)$ to be the set of *continuous* $c : X \rightarrow \mathbb{R}$, with operations Φ_f defined as in (2.1). For X a manifold with $\dim X > 0$, the C^∞ -rings $C^\infty(X)$ and $C^0(X)$ are different.

Definition 2.3. Let \mathfrak{C} be a C^∞ -ring. Then we may give \mathfrak{C} the structure of a *commutative \mathbb{R} -algebra*. Define addition $+$ on \mathfrak{C} by $c + c' = \Phi_f(c, c')$ for $c, c' \in \mathfrak{C}$, where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is $f(x, y) = x + y$. Define multiplication \cdot on \mathfrak{C} by $c \cdot c' = \Phi_g(c, c')$, where $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is $g(x, y) = xy$. Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c = \Phi_{\lambda'}(c)$, where $\lambda' : \mathbb{R} \rightarrow \mathbb{R}$ is $\lambda'(x) = \lambda x$. Define elements $0, 1 \in \mathfrak{C}$ by $0 = \Phi_{0'}(\emptyset)$ and $1 = \Phi_{1'}(\emptyset)$, where $0' : \mathbb{R}^0 \rightarrow \mathbb{R}$ and $1' : \mathbb{R}^0 \rightarrow \mathbb{R}$ are the maps $0' : \emptyset \mapsto 0$ and $1' : \emptyset \mapsto 1$. One can show using the relations on the Φ_f that the axioms of a commutative \mathbb{R} -algebra are satisfied. In Example 2.2, this yields the obvious \mathbb{R} -algebra structure on the smooth functions $c : X \rightarrow \mathbb{R}$.

An *ideal* I in \mathfrak{C} is an ideal $I \subset \mathfrak{C}$ in \mathfrak{C} regarded as a commutative \mathbb{R} -algebra. Then we make the quotient \mathfrak{C}/I into a C^∞ -ring as follows. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is

smooth, define $\Phi_f^I : (\mathfrak{C}/I)^n \rightarrow \mathfrak{C}/I$ by

$$(\Phi_f^I(c_1 + I, \dots, c_n + I))(x) = f(c_1(x), \dots, c_n(x)) + I.$$

Using Hadamard's Lemma, one can show that this is independent of the choice of representatives c_1, \dots, c_n . Then $(\mathfrak{C}/I, (\Phi_f^I)_{f: \mathbb{R}^n \rightarrow \mathbb{R}} C^\infty)$ is a C^∞ -ring.

A C^∞ -ring \mathfrak{C} is called *finitely generated* if there exist c_1, \dots, c_n in \mathfrak{C} which generate \mathfrak{C} over all C^∞ -operations. That is, for each $c \in \mathfrak{C}$ there exists smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $c = \Phi_f(c_1, \dots, c_n)$. Given such $\mathfrak{C}, c_1, \dots, c_n$, define $\phi : C^\infty(\mathbb{R}^n) \rightarrow \mathfrak{C}$ by $\phi(f) = \Phi_f(c_1, \dots, c_n)$ for smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$, where $C^\infty(\mathbb{R}^n)$ is as in Example 2.2 with $X = \mathbb{R}^n$. Then ϕ is a surjective morphism of C^∞ -rings, so $I = \text{Ker } \phi$ is an ideal in $C^\infty(\mathbb{R}^n)$, and $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ as a C^∞ -ring. Thus, \mathfrak{C} is finitely generated if and only if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geq 0$ and some ideal I in $C^\infty(\mathbb{R}^n)$.

2.2 C^∞ -schemes

Next we summarize material in [19, §4] on C^∞ -schemes.

Definition 2.4. A C^∞ -ringed space $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf \mathcal{O}_X of C^∞ -rings on X .

A *morphism* $\underline{f} = (f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of C^∞ ringed spaces is a continuous map $f : X \rightarrow Y$ and a morphism $f^\# : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$ of sheaves of C^∞ -rings on X , where $f^{-1}(\mathcal{O}_Y)$ is the inverse image sheaf. There is another way to write the data $f^\#$: since direct image of sheaves f_* is right adjoint to inverse image f^{-1} , there is a natural bijection

$$\text{Hom}_X(f^{-1}(\mathcal{O}_Y), \mathcal{O}_X) \cong \text{Hom}_Y(\mathcal{O}_Y, f_*(\mathcal{O}_X)). \quad (2.2)$$

Write $f_\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ for the morphism of sheaves of C^∞ -rings on Y corresponding to $f^\#$ under (2.2), so that

$$f^\# : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X \quad \longleftrightarrow \quad f_\# : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X). \quad (2.3)$$

Depending on the application, either $f^\#$ or $f_\#$ may be more useful. We choose to regard $f^\#$ as primary and write morphisms as $\underline{f} = (f, f^\#)$ rather than $(f, f_\#)$, because we find it convenient in [21] to work uniformly using pullbacks, rather than mixing pullbacks and pushforwards.

Write $\mathbf{C}^\infty\mathbf{RS}$ for the category of C^∞ -ringed spaces. As in [9, Th. 8] there is a *spectrum functor* $\text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{C}^\infty\mathbf{RS}$, defined explicitly in [19, Def. 4.12]. A C^∞ -ringed space \underline{X} is called an *affine C^∞ -scheme* if it is isomorphic in $\mathbf{C}^\infty\mathbf{RS}$ to $\text{Spec } \mathfrak{C}$ for some C^∞ -ring \mathfrak{C} . A C^∞ -ringed space $\underline{X} = (X, \mathcal{O}_X)$ is called a *C^∞ -scheme* if X can be covered by open sets $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine C^∞ -scheme. Write $\mathbf{C}^\infty\mathbf{Sch}$ for the full subcategory of C^∞ -schemes in $\mathbf{C}^\infty\mathbf{RS}$.

A C^∞ -scheme $\underline{X} = (X, \mathcal{O}_X)$ is called *locally fair* if X can be covered by open $U \subseteq X$ with $(U, \mathcal{O}_X|_U) \cong \text{Spec } \mathfrak{C}$ for some finitely generated C^∞ -ring

℄. Roughly speaking this means that \underline{X} is locally finite-dimensional. Write $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}$ for the full subcategory of locally fair C^∞ -schemes in $\mathbf{C}^\infty\mathbf{Sch}$.

We call a C^∞ -scheme \underline{X} *separated*, *second countable*, *compact*, or *paracompact*, if the underlying topological space X is Hausdorff, second countable, compact, or paracompact, respectively.

We define a C^∞ -scheme \underline{X} for each manifold X .

Example 2.5. Let X be a manifold. Define a C^∞ -ringed space $\underline{X} = (X, \mathcal{O}_X)$ to have topological space X and $\mathcal{O}_X(U) = C^\infty(U)$ for each open $U \subseteq X$, where $C^\infty(U)$ is the C^∞ -ring of smooth maps $c : U \rightarrow \mathbb{R}$, and if $V \subseteq U \subseteq X$ are open define $\rho_{UV} : C^\infty(U) \rightarrow C^\infty(V)$ by $\rho_{UV} : c \mapsto c|_V$. Then $\underline{X} = (X, \mathcal{O}_X)$ is a local C^∞ -ringed space. It is canonically isomorphic to $\text{Spec } C^\infty(X)$, and so is an affine C^∞ -scheme. It is locally fair.

Define a functor $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}} : \mathbf{Man} \rightarrow \mathbf{C}^\infty\mathbf{Sch}^{\text{lf}} \subset \mathbf{C}^\infty\mathbf{Sch}$ by $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}} = \text{Spec} \circ F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Rings}}$. Then $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}$ is full and faithful, and embeds \mathbf{Man} as a full subcategory of $\mathbf{C}^\infty\mathbf{Sch}$.

By [19, Cor. 4.21 & Th. 4.33] we have:

Theorem 2.6. *Fibre products and all finite limits exist in the category $\mathbf{C}^\infty\mathbf{Sch}$. The subcategory $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}$ is closed under fibre products and all finite limits in $\mathbf{C}^\infty\mathbf{Sch}$. The functor $F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}$ takes transverse fibre products in \mathbf{Man} to fibre products in $\mathbf{C}^\infty\mathbf{Sch}$.*

The proof of the existence of fibre products in $\mathbf{C}^\infty\mathbf{Sch}$ follows that for fibre products of schemes in Hartshorne [12, Th. II.3.3], together with the existence of C^∞ -scheme products $\underline{X} \times \underline{Y}$ of affine C^∞ -schemes $\underline{X}, \underline{Y}$. The latter follows from the existence of coproducts $\mathfrak{C} \hat{\otimes} \mathfrak{D}$ in $\mathbf{C}^\infty\mathbf{Rings}$ of C^∞ -rings $\mathfrak{C}, \mathfrak{D}$. Here $\mathfrak{C} \hat{\otimes} \mathfrak{D}$ may be thought of as a ‘completed tensor product’ of $\mathfrak{C}, \mathfrak{D}$. The actual tensor product $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D}$ is naturally an \mathbb{R} -algebra but not a C^∞ -ring, with an inclusion of \mathbb{R} -algebras $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D} \hookrightarrow \mathfrak{C} \hat{\otimes} \mathfrak{D}$, but $\mathfrak{C} \hat{\otimes} \mathfrak{D}$ is often much larger than $\mathfrak{C} \otimes_{\mathbb{R}} \mathfrak{D}$. For free C^∞ -rings we have $C^\infty(\mathbb{R}^m) \hat{\otimes} C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^{m+n})$.

In [19, Def. 4.34 & Prop. 4.35] we discuss *partitions of unity* on C^∞ -schemes.

Definition 2.7. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞ -scheme. Consider a formal sum $\sum_{a \in A} c_a$, where A is an indexing set and $c_a \in \mathcal{O}_X(X)$ for $a \in A$. We say $\sum_{a \in A} c_a$ is a *locally finite sum on \underline{X}* if X can be covered by open $U \subseteq X$ such that for all but finitely many $a \in A$ we have $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$.

By the sheaf axioms for \mathcal{O}_X , if $\sum_{a \in A} c_a$ is a locally finite sum there exists a unique $c \in \mathcal{O}_X(X)$ such that for all open $U \subseteq X$ with $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$ for all but finitely many $a \in A$, we have $\rho_{XU}(c) = \sum_{a \in A} \rho_{XU}(c_a)$ in $\mathcal{O}_X(U)$, where the sum makes sense as there are only finitely many nonzero terms. We call c the *limit* of $\sum_{a \in A} c_a$, written $\sum_{a \in A} c_a = c$.

Let $c \in \mathcal{O}_X(X)$. Suppose $V_i \subseteq X$ is open and $\rho_{XV_i}(c) = 0 \in \mathcal{O}_X(V_i)$ for $i \in I$, and let $V = \bigcup_{i \in I} V_i$. Then $V \subseteq X$ is open, and $\rho_{XV}(c) = 0 \in \mathcal{O}_X(V)$ as \mathcal{O}_X is a sheaf. Thus taking the union of all open $V \subseteq X$ with $\rho_{XV}(c) = 0$ gives a unique maximal open set $V_c \subseteq X$ such that $\rho_{XV_c}(c) = 0 \in \mathcal{O}_X(V_c)$. Define

the *support* $\text{supp } c$ of c to be $X \setminus V_c$, so that $\text{supp } c$ is closed in X . If $U \subseteq X$ is open, we say that c is *supported in* U if $\text{supp } c \subseteq U$.

Let $\{U_a : a \in A\}$ be an open cover of X . A *partition of unity on \underline{X} subordinate to $\{U_a : a \in A\}$* is $\{\eta_a : a \in A\}$ with $\eta_a \in \mathcal{O}_X(X)$ supported on U_a for $a \in A$, such that $\sum_{a \in A} \eta_a$ is a locally finite sum on \underline{X} with $\sum_{a \in A} \eta_a = 1$.

Proposition 2.8. *Suppose \underline{X} is a separated, paracompact, locally fair C^∞ -scheme, and $\{\underline{U}_a : a \in A\}$ an open cover of \underline{X} . Then there exists a partition of unity $\{\eta_a : a \in A\}$ on \underline{X} subordinate to $\{\underline{U}_a : a \in A\}$.*

Here are some differences between ordinary schemes and C^∞ -schemes:

Remark 2.9. (i) If A is a ring or algebra, then points of the corresponding scheme $\text{Spec } A$ are prime ideals in A . However, if \mathfrak{C} is a C^∞ -ring then (by definition) points of $\text{Spec } \mathfrak{C}$ are maximal ideals in \mathfrak{C} with residue field \mathbb{R} , or equivalently, \mathbb{R} -algebra morphisms $x : \mathfrak{C} \rightarrow \mathbb{R}$. This has the effect that if X is a manifold then points of $\text{Spec } C^\infty(X)$ are just points of X .

(ii) In conventional algebraic geometry, affine schemes are a restrictive class. Central examples such as \mathbb{CP}^n are not affine, and affine schemes are not closed under open subsets, so that \mathbb{C}^2 is affine but $\mathbb{C}^2 \setminus \{0\}$ is not. In contrast, affine C^∞ -schemes are already general enough for many purposes. For example:

- All manifolds are fair affine C^∞ -schemes.
- Open C^∞ -subschemes of fair affine C^∞ -schemes are fair and affine.
- Separated, second countable, locally fair C^∞ -schemes are affine.

Affine C^∞ -schemes are always separated (Hausdorff), so we need general C^∞ -schemes to include non-Hausdorff behaviour.

(iii) In conventional algebraic geometry the Zariski topology is too coarse for many purposes, so one has to introduce the étale topology. In C^∞ -algebraic geometry there is no need for this, as affine C^∞ -schemes are Hausdorff.

(iv) Even very basic C^∞ -rings such as $C^\infty(\mathbb{R}^n)$ for $n > 0$ are not noetherian as \mathbb{R} -algebras. So C^∞ -schemes should be compared to non-noetherian schemes in conventional algebraic geometry.

(v) The existence of partitions of unity, as in Proposition 2.8, makes some things easier in C^∞ -algebraic geometry than in conventional algebraic geometry. For example, geometric objects can often be ‘glued together’ over the subsets of an open cover using partitions of unity, and if \mathcal{E} is a quasicoherent sheaf on a separated, paracompact, locally fair C^∞ -scheme \underline{X} then $H^i(\mathcal{E}) = 0$ for $i > 0$.

2.3 Modules over C^∞ -rings, and cotangent modules

In [19, §5] we discuss modules over C^∞ -rings.

Definition 2.10. Let \mathfrak{C} be a C^∞ -ring. A \mathfrak{C} -*module* M is a module over \mathfrak{C} regarded as a commutative \mathbb{R} -algebra as in Definition 2.3. \mathfrak{C} -modules form an abelian category, which we write as $\mathfrak{C}\text{-mod}$. For example, \mathfrak{C} is a \mathfrak{C} -module, and

more generally $\mathfrak{C} \otimes_{\mathbb{R}} V$ is a \mathfrak{C} -module for any real vector space V . Let $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ be a morphism of C^∞ -rings. If M is a \mathfrak{C} -module then $\phi_*(M) = M \otimes_{\mathfrak{C}} \mathfrak{D}$ is a \mathfrak{D} -module. This induces a functor $\phi_* : \mathfrak{C}\text{-mod} \rightarrow \mathfrak{D}\text{-mod}$.

Example 2.11. Let X be a manifold, and $E \rightarrow X$ a vector bundle. Write $C^\infty(E)$ for the vector space of smooth sections e of E . Then $C^\infty(X)$ acts on $C^\infty(E)$ by multiplication, so $C^\infty(E)$ is a $C^\infty(X)$ -module.

In [19, §5.3] we define the *cotangent module* $\Omega_{\mathfrak{C}}$ of a C^∞ -ring \mathfrak{C} .

Definition 2.12. Let \mathfrak{C} be a C^∞ -ring, and M a \mathfrak{C} -module. A C^∞ -derivation is an \mathbb{R} -linear map $d : \mathfrak{C} \rightarrow M$ such that whenever $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth map and $c_1, \dots, c_n \in \mathfrak{C}$, we have

$$d\Phi_f(c_1, \dots, c_n) = \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_n) \cdot dc_i.$$

We call such a pair M, d a *cotangent module* for \mathfrak{C} if it has the universal property that for any \mathfrak{C} -module M' and C^∞ -derivation $d' : \mathfrak{C} \rightarrow M'$, there exists a unique morphism of \mathfrak{C} -modules $\phi : M \rightarrow M'$ with $d' = \phi \circ d$.

Define $\Omega_{\mathfrak{C}}$ to be the quotient of the free \mathfrak{C} -module with basis of symbols dc for $c \in \mathfrak{C}$ by the \mathfrak{C} -submodule spanned by all expressions of the form $d\Phi_f(c_1, \dots, c_n) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_n) \cdot dc_i$ for $f : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth and $c_1, \dots, c_n \in \mathfrak{C}$, and define $d_{\mathfrak{C}} : \mathfrak{C} \rightarrow \Omega_{\mathfrak{C}}$ by $d_{\mathfrak{C}} : c \mapsto dc$. Then $\Omega_{\mathfrak{C}}, d_{\mathfrak{C}}$ is a cotangent module for \mathfrak{C} . Thus cotangent modules always exist, and are unique up to unique isomorphism.

Let $\mathfrak{C}, \mathfrak{D}$ be C^∞ -rings with cotangent modules $\Omega_{\mathfrak{C}}, d_{\mathfrak{C}}, \Omega_{\mathfrak{D}}, d_{\mathfrak{D}}$, and $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ be a morphism of C^∞ -rings. Then ϕ makes $\Omega_{\mathfrak{D}}$ into a \mathfrak{C} -module, and there is a unique morphism $\Omega_{\phi} : \Omega_{\mathfrak{C}} \rightarrow \Omega_{\mathfrak{D}}$ in $\mathfrak{C}\text{-mod}$ with $d_{\mathfrak{D}} \circ \phi = \Omega_{\phi} \circ d_{\mathfrak{C}}$. This induces a morphism $(\Omega_{\phi})_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \rightarrow \Omega_{\mathfrak{D}}$ in $\mathfrak{D}\text{-mod}$ with $(\Omega_{\phi})_* \circ (d_{\mathfrak{C}} \otimes \text{id}_{\mathfrak{D}}) = d_{\mathfrak{D}}$.

Example 2.13. Let X be a manifold. Then the cotangent bundle T^*X is a vector bundle over X , so as in Example 2.11 it yields a $C^\infty(X)$ -module $C^\infty(T^*X)$. The exterior derivative $d : C^\infty(X) \rightarrow C^\infty(T^*X)$ is a C^∞ -derivation. These $C^\infty(T^*X), d$ have the universal property in Definition 2.12, and so form a *cotangent module* for $C^\infty(X)$.

Now let X, Y be manifolds, and $f : X \rightarrow Y$ be smooth. Then $f^*(TY), TX$ are vector bundles over X , and the derivative of f is a vector bundle morphism $df : TX \rightarrow f^*(TY)$. The dual of this morphism is $df^* : f^*(T^*Y) \rightarrow T^*X$. This induces a morphism of $C^\infty(X)$ -modules $(df^*)_* : C^\infty(f^*(T^*Y)) \rightarrow C^\infty(T^*X)$. This $(df^*)_*$ is identified with $(\Omega_{f^*})_*$ in Definition 2.12 under the natural isomorphism $C^\infty(f^*(T^*Y)) \cong C^\infty(T^*Y) \otimes_{C^\infty(Y)} C^\infty(X)$.

Definition 2.12 abstracts the notion of cotangent bundle of a manifold in a way that makes sense for any C^∞ -ring.

2.4 Quasicoherent sheaves on C^∞ -schemes

In [19, §6] we discuss sheaves of modules on C^∞ -schemes.

Definition 2.14. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞ -scheme. An \mathcal{O}_X -module \mathcal{E} on \underline{X} assigns a module $\mathcal{E}(U)$ over $\mathcal{O}_X(U)$ for each open set $U \subseteq X$, with $\mathcal{O}_X(U)$ -action $\mu_U : \mathcal{O}_X(U) \times \mathcal{E}(U) \rightarrow \mathcal{E}(U)$, and a linear map $\mathcal{E}_{UV} : \mathcal{E}(U) \rightarrow \mathcal{E}(V)$ for each inclusion of open sets $V \subseteq U \subseteq X$, such that the following commutes:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{E}(U) & \xrightarrow{\mu_U} & \mathcal{E}(U) \\ \downarrow \rho_{UV} \times \mathcal{E}_{UV} & & \mathcal{E}_{UV} \downarrow \\ \mathcal{O}_X(V) \times \mathcal{E}(V) & \xrightarrow{\mu_V} & \mathcal{E}(V), \end{array}$$

and all this data $\mathcal{E}(U), \mathcal{E}_{UV}$ satisfies the usual sheaf axioms [12, §II.1].

A *morphism of \mathcal{O}_X -modules* $\phi : \mathcal{E} \rightarrow \mathcal{F}$ assigns a morphism of $\mathcal{O}_X(U)$ -modules $\phi(U) : \mathcal{E}(U) \rightarrow \mathcal{F}(U)$ for each open set $U \subseteq X$, such that $\phi(V) \circ \mathcal{E}_{UV} = \mathcal{F}_{UV} \circ \phi(U)$ for each inclusion of open sets $V \subseteq U \subseteq X$. Then \mathcal{O}_X -modules form an abelian category, which we write as $\mathcal{O}_X\text{-mod}$.

As in [19, §6.2], the spectrum functor $\text{Spec} : \mathbf{C}^\infty\mathbf{Rings}^{\text{op}} \rightarrow \mathbf{C}^\infty\mathbf{Sch}$ has a counterpart for modules: if \mathfrak{C} is a C^∞ -ring and $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$ we can define a functor $\text{MSpec} : \mathfrak{C}\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$. If \mathfrak{C} is a *fair* C^∞ -ring, there is a full abelian subcategory $\mathfrak{C}\text{-mod}^{\text{co}}$ of *complete* \mathfrak{C} -modules in $\mathfrak{C}\text{-mod}$, such that $\text{MSpec}|_{\mathfrak{C}\text{-mod}^{\text{co}}} : \mathfrak{C}\text{-mod}^{\text{co}} \rightarrow \mathcal{O}_X\text{-mod}$ is an equivalence of categories, with quasi-inverse the global sections functor $\Gamma : \mathcal{O}_X\text{-mod} \rightarrow \mathfrak{C}\text{-mod}^{\text{co}}$. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞ -scheme, and \mathcal{E} an \mathcal{O}_X -module. We call \mathcal{E} *quasicoherent* if \underline{X} can be covered by open \underline{U} with $\underline{U} \cong \text{Spec } \mathfrak{C}$ for some C^∞ -ring \mathfrak{C} , and under this identification $\mathcal{E}|_{\underline{U}} \cong \text{MSpec } M$ for some \mathfrak{C} -module M . We call \mathcal{E} a *vector bundle of rank* $n \geq 0$ if \underline{X} may be covered by open \underline{U} such that $\mathcal{E}|_{\underline{U}} \cong \mathcal{O}_U \otimes_{\mathbb{R}} \mathbb{R}^n$.

Write $\text{qcoh}(\underline{X}), \text{vect}(\underline{X})$ for the full subcategories of quasicoherent sheaves and vector bundles in $\mathcal{O}_X\text{-mod}$. Then $\text{qcoh}(\underline{X})$ is an abelian category. Since $\text{MSpec} : \mathfrak{C}\text{-mod}^{\text{co}} \rightarrow \mathcal{O}_X\text{-mod}$ is an equivalence for \mathfrak{C} fair and $(X, \mathcal{O}_X) = \text{Spec } \mathfrak{C}$, as in [19, Cor. 6.11] we see that if \underline{X} is a locally fair C^∞ -scheme then every \mathcal{O}_X -module \mathcal{E} on \underline{X} is quasicoherent, that is, $\text{qcoh}(\underline{X}) = \mathcal{O}_X\text{-mod}$.

Remark 2.15. If \underline{X} is a separated, paracompact, locally fair C^∞ -scheme then vector bundles on \underline{X} are projective objects in the abelian category $\text{qcoh}(\underline{X})$.

Definition 2.16. Let $\underline{f} : \underline{X} \rightarrow \underline{Y}$ be a morphism of C^∞ -schemes, and let \mathcal{E} be an \mathcal{O}_Y -module. Define the *pullback* $\underline{f}^*(\mathcal{E})$, an \mathcal{O}_X -module, by $\underline{f}^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $f^{-1}(\mathcal{E}), f^{-1}(\mathcal{O}_Y)$ are inverse image sheaves, and the tensor product uses the morphism $f^\# : f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$. If $\phi : \mathcal{E} \rightarrow \mathcal{F}$ is a morphism in $\mathcal{O}_Y\text{-mod}$ we have an induced morphism $\underline{f}^*(\phi) = f^{-1}(\phi) \otimes \text{id}_{\mathcal{O}_X} : \underline{f}^*(\mathcal{E}) \rightarrow \underline{f}^*(\mathcal{F})$ in $\mathcal{O}_X\text{-mod}$. Then $\underline{f}^* : \mathcal{O}_Y\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$ is a right exact functor, which restricts to a right exact functor $\underline{f}^* : \text{qcoh}(\underline{Y}) \rightarrow \text{qcoh}(\underline{X})$.

Remark 2.17. Pullbacks $\underline{f}^*(\mathcal{E})$ are characterized by a universal property, and so are unique up to canonical isomorphism, rather than unique. Our definition of $\underline{f}^*(\mathcal{E})$ is not functorial in \underline{f} . That is, if $\underline{f} : \underline{X} \rightarrow \underline{Y}, \underline{g} : \underline{Y} \rightarrow \underline{Z}$ are morphisms and $\mathcal{E} \in \mathcal{O}_Z\text{-mod}$ then $(\underline{g} \circ \underline{f})^*(\mathcal{E})$ and $\underline{f}^*(\underline{g}^*(\mathcal{E}))$ are canonically isomorphic in $\mathcal{O}_X\text{-mod}$, but may not be equal. In [21] we keep track of these canonical isomorphisms, writing them as $I_{\underline{f}, \underline{g}}(\mathcal{E}) : (\underline{g} \circ \underline{f})^*(\mathcal{E}) \rightarrow \underline{f}^*(\underline{g}^*(\mathcal{E}))$. However, in

this survey, by an abuse of notation that is common in the literature, we will for simplicity omit the isomorphisms $I_{f,g}(\mathcal{E})$, and identify $(g \circ f)^*(\mathcal{E})$ with $f^*(g^*(\mathcal{E}))$.

Similarly, when \underline{f} is the identity $\underline{\text{id}}_{\underline{X}} : \underline{X} \rightarrow \underline{X}$ and $\underline{\mathcal{E}} \in \mathcal{O}_{\underline{X}}\text{-mod}$, we may not have $\underline{\text{id}}_{\underline{X}}^*(\underline{\mathcal{E}}) = \underline{\mathcal{E}}$, but there is a canonical isomorphism $\delta_{\underline{X}}(\underline{\mathcal{E}}) : \underline{\text{id}}_{\underline{X}}^*(\underline{\mathcal{E}}) \rightarrow \underline{\mathcal{E}}$, which we keep track of in [21]. But here, for simplicity, by an abuse of notation we omit $\delta_{\underline{X}}(\underline{\mathcal{E}})$, and identify $\underline{\text{id}}_{\underline{X}}^*(\underline{\mathcal{E}})$ with $\underline{\mathcal{E}}$.

Example 2.18. Let X be a manifold, and \underline{X} the associated C^∞ -scheme from Example 2.5, so that $\mathcal{O}_{\underline{X}}(U) = C^\infty(U)$ for all open $U \subseteq X$. Let $E \rightarrow X$ be a vector bundle. Define an $\mathcal{O}_{\underline{X}}$ -module \mathcal{E} on \underline{X} by $\mathcal{E}(U) = C^\infty(E|_U)$, the smooth sections of the vector bundle $E|_U \rightarrow U$, and for open $V \subseteq U \subseteq X$ define $\mathcal{E}_{UV} : \mathcal{E}(U) \rightarrow \mathcal{E}(V)$ by $\mathcal{E}_{UV} : e_U \mapsto e_U|_V$. Then $\mathcal{E} \in \text{vect}(\underline{X})$ is a vector bundle on \underline{X} , which we think of as a lift of E from manifolds to C^∞ -schemes.

Let $f : X \rightarrow Y$ be a smooth map of manifolds, and $\underline{f} : \underline{X} \rightarrow \underline{Y}$ the corresponding morphism of C^∞ -schemes. Let $F \rightarrow Y$ be a vector bundle over Y , so that $f^*(F) \rightarrow X$ is a vector bundle over X . Let $\mathcal{F} \in \text{vect}(\underline{Y})$ be the vector bundle over \underline{Y} lifting F . Then $\underline{f}^*(\mathcal{F})$ is canonically isomorphic to the vector bundle over \underline{X} lifting $f^*(F)$.

We define *cotangent sheaves*, the sheaf version of cotangent modules in §2.3.

Definition 2.19. Let \underline{X} be a C^∞ -scheme. Define $\mathcal{PT}^*\underline{X}$ to associate to each open $U \subseteq X$ the cotangent module $\Omega_{\mathcal{O}_X(U)}$, and to each inclusion of open sets $V \subseteq U \subseteq X$ the morphism of $\mathcal{O}_X(U)$ -modules $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \rightarrow \Omega_{\mathcal{O}_X(V)}$ associated to the morphism of C^∞ -rings $\rho_{UV} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$. Then $\mathcal{PT}^*\underline{X}$ is a *presheaf of \mathcal{O}_X -modules on \underline{X}* . Define the *cotangent sheaf $T^*\underline{X}$ of \underline{X}* to be the sheafification of $\mathcal{PT}^*\underline{X}$, as an \mathcal{O}_X -module.

Let $\underline{f} : \underline{X} \rightarrow \underline{Y}$ be a morphism of C^∞ -schemes. Then by Definition 2.16, $\underline{f}^*(T^*\underline{Y}) = f^{-1}(T^*\underline{Y}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $T^*\underline{Y}$ is the sheafification of the presheaf $V \mapsto \Omega_{\mathcal{O}_Y(V)}$, and $f^{-1}(T^*\underline{Y})$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)} (T^*\underline{Y})(V)$, and $f^{-1}(\mathcal{O}_Y)$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{O}_Y(V)$. The three sheafifications combine into one, so that $\underline{f}^*(T^*\underline{Y})$ is the sheafification of the presheaf $\mathcal{P}(\underline{f}^*(T^*\underline{Y}))$ acting by

$$U \mapsto \mathcal{P}(\underline{f}^*(T^*\underline{Y}))(U) = \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U).$$

Define a morphism of presheaves $\mathcal{P}\Omega_{\underline{f}} : \mathcal{P}(\underline{f}^*(T^*\underline{Y})) \rightarrow \mathcal{PT}^*\underline{X}$ on X by

$$(\mathcal{P}\Omega_{\underline{f}})(U) = \lim_{V \supseteq f(U)} (\Omega_{\rho_{f^{-1}(V)U} \circ f_{\sharp}(V)})^*,$$

where $(\Omega_{\rho_{f^{-1}(V)U} \circ f_{\sharp}(V)})^* : \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \rightarrow \Omega_{\mathcal{O}_X(U)} = (\mathcal{PT}^*\underline{X})(U)$ is constructed as in Definition 2.12 from the C^∞ -ring morphisms $f_{\sharp}(V) : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V))$ from $f_{\sharp} : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ corresponding to f_{\sharp} in \underline{f} as in (2.3), and $\rho_{f^{-1}(V)U} : \mathcal{O}_X(f^{-1}(V)) \rightarrow \mathcal{O}_X(U)$ in \mathcal{O}_X . Define $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \rightarrow T^*\underline{X}$ to be the induced morphism of the associated sheaves.

Example 2.20. Let X be a manifold, and \underline{X} the associated C^∞ -scheme. Then $T^*\underline{X}$ is a vector bundle on \underline{X} , and is canonically isomorphic to the lift to C^∞ -schemes from Example 2.18 of the cotangent vector bundle T^*X of X .

Here [19, Th. 6.17] are some properties of cotangent sheaves.

Theorem 2.21. (a) Let $\underline{f} : \underline{X} \rightarrow \underline{Y}$ and $\underline{g} : \underline{Y} \rightarrow \underline{Z}$ be morphisms of C^∞ -schemes. Then

$$\Omega_{\underline{g} \circ \underline{f}} = \Omega_{\underline{f}} \circ \underline{f}^*(\Omega_{\underline{g}})$$

as morphisms $(\underline{g} \circ \underline{f})^*(T^*\underline{Z}) \rightarrow T^*\underline{X}$ in $\mathcal{O}_X\text{-mod}$. Here $\Omega_{\underline{g}} : \underline{g}^*(T^*\underline{Z}) \rightarrow T^*\underline{Y}$ is a morphism in $\mathcal{O}_Y\text{-mod}$, so applying \underline{f}^* gives $\underline{f}^*(\Omega_{\underline{g}}) : (\underline{g} \circ \underline{f})^*(T^*\underline{Z}) = \underline{f}^*(\underline{g}^*(T^*\underline{Z})) \rightarrow \underline{f}^*(T^*\underline{Y})$ in $\mathcal{O}_X\text{-mod}$.

(b) Suppose $\underline{W}, \underline{X}, \underline{Y}, \underline{Z}$ are locally fair C^∞ -schemes with a Cartesian square

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\quad f \quad} & \underline{Y} \\ \downarrow \underline{e} & \searrow \underline{f} & \downarrow \underline{h} \\ \underline{X} & \xrightarrow{\quad g \quad} & \underline{Z} \end{array}$$

in $\mathbf{C}^\infty\mathbf{Sch}^{\text{lf}}$, so that $\underline{W} = \underline{X} \times_{\underline{Z}} \underline{Y}$. Then the following is exact in $\text{qcoh}(\underline{W})$:

$$(\underline{g} \circ \underline{e})^*(T^*\underline{Z}) \xrightarrow{\underline{e}^*(\Omega_{\underline{g}}) \oplus -\underline{f}^*(\Omega_{\underline{h}})} \underline{e}^*(T^*\underline{X}) \oplus \underline{f}^*(T^*\underline{Y}) \xrightarrow{\Omega_{\underline{e}} \oplus \Omega_{\underline{f}}} T^*\underline{W} \longrightarrow 0.$$

3 The 2-category of d-spaces

We will now define the 2-category of *d-spaces* \mathbf{dSpa} , following [21, §2]. D-spaces are ‘derived’ versions of C^∞ -schemes. In §4 we will define the 2-category of d-manifolds \mathbf{dMan} as a 2-subcategory of \mathbf{dSpa} . For an introduction to 2-categories, see Appendix A.

3.1 The definition of d-spaces

Definition 3.1. A *d-space* \mathbf{X} is a quintuple $\mathbf{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X)$ such that $\underline{X} = (X, \mathcal{O}_X)$ is a separated, second countable, locally fair C^∞ -scheme, and $\mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X$ fit into an exact sequence of sheaves on X

$$\mathcal{E}_X \xrightarrow{j_X} \mathcal{O}'_X \xrightarrow{\iota_X} \mathcal{O}_X \longrightarrow 0,$$

satisfying the conditions:

- (a) \mathcal{O}'_X is a sheaf of C^∞ -rings on X , with $\underline{X}' = (X, \mathcal{O}'_X)$ a C^∞ -scheme.
- (b) $\iota_X : \mathcal{O}'_X \rightarrow \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞ -rings on X . Its kernel \mathcal{I}_X is a sheaf of ideals in \mathcal{O}'_X , which should be a sheaf of square zero ideals. Here a *square zero ideal* in a commutative \mathbb{R} -algebra A is an ideal I with $i \cdot j = 0$ for all $i, j \in I$. Then \mathcal{I}_X is an \mathcal{O}'_X -module, but as \mathcal{I}_X consists of square zero ideals and ι_X is surjective, the \mathcal{O}'_X -action factors through an \mathcal{O}_X -action. Hence \mathcal{I}_X is an \mathcal{O}_X -module, and thus a quasicoherent sheaf on \underline{X} , as \underline{X} is locally fair.

- (c) \mathcal{E}_X is a quasicoherent sheaf on \underline{X} , and $j_X : \mathcal{E}_X \rightarrow \mathcal{I}_X$ is a surjective morphism in $\text{qcoh}(\underline{X})$.

As \underline{X} is locally fair, the underlying topological space X is locally homeomorphic to a closed subset of \mathbb{R}^n , so it is *locally compact*. But Hausdorff, second countable and locally compact imply paracompact, and thus \underline{X} is *paracompact*.

The sheaf of C^∞ -rings \mathcal{O}'_X has a sheaf of cotangent modules $\Omega_{\mathcal{O}'_X}$, which is an \mathcal{O}'_X -module with exterior derivative $d : \mathcal{O}'_X \rightarrow \Omega_{\mathcal{O}'_X}$. Define $\mathcal{F}_X = \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X$ to be the associated \mathcal{O}_X -module, a quasicoherent sheaf on \underline{X} , and set $\psi_X = \Omega_{\iota_X} \otimes \text{id} : \mathcal{F}_X \rightarrow T^*\underline{X}$, a morphism in $\text{qcoh}(\underline{X})$. Define $\phi_X : \mathcal{E}_X \rightarrow \mathcal{F}_X$ to be the composition of morphisms of sheaves of abelian groups on X :

$$\mathcal{E}_X \xrightarrow{j_X} \mathcal{I}_X \xrightarrow{d|_{\mathcal{I}_X}} \Omega_{\mathcal{O}'_X} \cong \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}'_X \xrightarrow{\text{id} \otimes \iota_X} \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X = \mathcal{F}_X.$$

It turns out that ϕ_X is actually a morphism of \mathcal{O}_X -modules, and the following sequence is exact in $\text{qcoh}(\underline{X})$:

$$\mathcal{E}_X \xrightarrow{\phi_X} \mathcal{F}_X \xrightarrow{\psi_X} T^*\underline{X} \longrightarrow 0.$$

The morphism $\phi_X : \mathcal{E}_X \rightarrow \mathcal{F}_X$ will be called the *virtual cotangent sheaf* of \mathbf{X} , for reasons we explain in §4.3.

Let \mathbf{X}, \mathbf{Y} be d-spaces. A 1-morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a triple $\mathbf{f} = (\underline{f}, f', f'')$, where $\underline{f} = (f, f^\sharp) : \underline{X} \rightarrow \underline{Y}$ is a morphism of C^∞ -schemes, $f' : f^{-1}(\mathcal{O}'_Y) \rightarrow \mathcal{O}'_X$ a morphism of sheaves of C^∞ -rings on X , and $f'' : f^*(\mathcal{E}_Y) \rightarrow \mathcal{E}_X$ a morphism in $\text{qcoh}(\underline{X})$, such that the following diagram of sheaves on X commutes:

$$\begin{array}{ccccccc} f^{-1}(\mathcal{E}_Y) \otimes_{f^{-1}(\mathcal{O}_Y)}^{\text{id}} f^{-1}(\mathcal{O}_Y) = f^{-1}(\mathcal{E}_Y) & \xrightarrow{f^{-1}(j_Y)} & f^{-1}(\mathcal{O}'_Y) & \xrightarrow{f^{-1}(\iota_Y)} & f^{-1}(\mathcal{O}_Y) & \longrightarrow & 0 \\ \downarrow \text{id} \otimes f^\sharp & & \downarrow f' & & \downarrow f^\sharp & & \\ \underline{f}^*(\mathcal{E}_Y) = f^{-1}(\mathcal{E}_Y) \otimes_{f^{-1}(\mathcal{O}_Y)}^{f^\sharp} \mathcal{O}_X & \xrightarrow{f''} & \mathcal{E}_X & \xrightarrow{j_X} & \mathcal{O}'_X & \xrightarrow{\iota_X} & \mathcal{O}_X \longrightarrow 0. \end{array}$$

Define morphisms $f^2 = \Omega_{f'} \otimes \text{id} : f^*(\mathcal{F}_Y) \rightarrow \mathcal{F}_X$ and $f^3 = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \rightarrow T^*\underline{X}$ in $\text{qcoh}(\underline{X})$. Then the following commutes in $\text{qcoh}(\underline{X})$, with exact rows:

$$\begin{array}{ccccccc} \underline{f}^*(\mathcal{E}_Y) & \xrightarrow{\underline{f}^*(\phi_Y)} & \underline{f}^*(\mathcal{F}_Y) & \xrightarrow{\underline{f}^*(\psi_Y)} & \underline{f}^*(T^*\underline{Y}) & \longrightarrow & 0 \\ \downarrow f'' & & \downarrow f^2 & & \downarrow f^3 & & \\ \mathcal{E}_X & \xrightarrow{\phi_X} & \mathcal{F}_X & \xrightarrow{\psi_X} & T^*\underline{X} & \longrightarrow & 0. \end{array} \quad (3.1)$$

If \mathbf{X} is a d-space, the *identity 1-morphism* $\text{id}_{\mathbf{X}} : \mathbf{X} \rightarrow \mathbf{X}$ is $\text{id}_{\mathbf{X}} = (\text{id}_{\underline{X}}, \text{id}_{\mathcal{O}'_X}, \text{id}_{\mathcal{E}_X})$. Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be d-spaces, and $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$, $\mathbf{g} : \mathbf{Y} \rightarrow \mathbf{Z}$ be 1-morphisms. Define the *composition of 1-morphisms* $\mathbf{g} \circ \mathbf{f} : \mathbf{X} \rightarrow \mathbf{Z}$ to be

$$\mathbf{g} \circ \mathbf{f} = (\underline{g} \circ \underline{f}, f' \circ f^{-1}(g'), f'' \circ \underline{f}^*(g'')).$$

Let $\mathbf{f}, \mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$ be 1-morphisms of d-spaces, where $\mathbf{f} = (f, f', f'')$ and $\mathbf{g} = (g, g', g'')$. Suppose $\underline{f} = \underline{g}$. A 2-morphism $\eta : \mathbf{f} \Rightarrow \mathbf{g}$ is a morphism $\eta : \underline{f}^*(\mathcal{F}_Y) \rightarrow \mathcal{E}_X$ in $\text{qcoh}(\underline{X})$, such that

$$\begin{aligned} g' &= f' + j_X \circ \eta \circ (\text{id} \otimes (f^\# \circ f^{-1}(\iota_Y))) \circ (f^{-1}(d)) \\ \text{and} \quad g'' &= f'' + \eta \circ \underline{f}^*(\phi_Y). \end{aligned}$$

Then $g^2 = f^2 + \phi_X \circ \eta$ and $g^3 = f^3$, so (3.1) for \mathbf{f}, \mathbf{g} combine to give a diagram

$$\begin{array}{ccccccc} \underline{f}^*(\mathcal{E}_Y) & \xrightarrow{\underline{f}^*(\phi_Y)} & \underline{f}^*(\mathcal{F}_Y) & \xrightarrow{\underline{f}^*(\psi_Y)} & \underline{f}^*(T^*\underline{Y}) & \longrightarrow & 0 \\ \downarrow \underline{f}'' & \swarrow g'' = f'' + \eta \circ \underline{f}^*(\phi_Y) & \downarrow f^2 & \swarrow g^2 = f^2 + \phi_X \circ \eta & \downarrow f^3 = g^3 & & \\ \mathcal{E}_X & \xleftarrow{\phi_X} & \mathcal{F}_X & \xrightarrow{\psi_X} & T^*\underline{X} & \longrightarrow & 0. \end{array} \quad (3.2)$$

That is, η is a homotopy between the morphisms of complexes (3.1) from \mathbf{f}, \mathbf{g} .

If $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a 1-morphism, the *identity 2-morphism* $\text{id}_{\mathbf{f}} : \mathbf{f} \Rightarrow \mathbf{f}$ is the zero morphism $0 : \underline{f}^*(\mathcal{F}_Y) \rightarrow \mathcal{E}_X$. Suppose \mathbf{X}, \mathbf{Y} are d-spaces, $\mathbf{f}, \mathbf{g}, \mathbf{h} : \mathbf{X} \rightarrow \mathbf{Y}$ are 1-morphisms and $\eta : \mathbf{f} \Rightarrow \mathbf{g}$, $\zeta : \mathbf{g} \Rightarrow \mathbf{h}$ are 2-morphisms. The *vertical composition of 2-morphisms* $\zeta \odot \eta : \mathbf{f} \Rightarrow \mathbf{h}$ as in (A.1) is $\zeta \odot \eta = \zeta + \eta$.

Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be d-spaces, $\mathbf{f}, \tilde{\mathbf{f}} : \mathbf{X} \rightarrow \mathbf{Y}$ and $\mathbf{g}, \tilde{\mathbf{g}} : \mathbf{Y} \rightarrow \mathbf{Z}$ be 1-morphisms, and $\eta : \mathbf{f} \Rightarrow \tilde{\mathbf{f}}$, $\zeta : \mathbf{g} \Rightarrow \tilde{\mathbf{g}}$ be 2-morphisms. The *horizontal composition of 2-morphisms* $\zeta * \eta : \mathbf{g} \circ \mathbf{f} \Rightarrow \tilde{\mathbf{g}} \circ \tilde{\mathbf{f}}$ as in (A.2) is

$$\zeta * \eta = \eta \circ \underline{f}^*(g^2) + f'' \circ \underline{f}^*(\zeta) + \eta \circ \underline{f}^*(\phi_Y) \circ \underline{f}^*(\zeta).$$

Regard the category $\mathbf{C}^\infty \mathbf{Sch}_{\text{ssc}}^{\text{lf}}$ of separated, second countable, locally fair C^∞ -schemes as a 2-category with only identity 2-morphisms $\text{id}_{\underline{f}}$ for (1-)morphisms $\underline{f} : \underline{X} \rightarrow \underline{Y}$. Define a 2-functor $F_{\mathbf{C}^\infty \mathbf{Sch}}^{\mathbf{dSpa}} : \mathbf{C}^\infty \mathbf{Sch}_{\text{ssc}}^{\text{lf}} \rightarrow \mathbf{dSpa}$ to map \underline{X} to $\mathbf{X} = (\underline{X}, \mathcal{O}_X, 0, \text{id}_{\mathcal{O}_X}, 0)$ on objects \underline{X} , to map \underline{f} to $\mathbf{f} = (\underline{f}, f^\#, 0)$ on (1-)morphisms $\underline{f} : \underline{X} \rightarrow \underline{Y}$, and to map identity 2-morphisms $\text{id}_{\underline{f}} : \underline{f} \Rightarrow \underline{f}$ to identity 2-morphisms $\text{id}_{\mathbf{f}} : \mathbf{f} \Rightarrow \mathbf{f}$. Define a 2-functor $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \rightarrow \mathbf{dSpa}$ by $F_{\mathbf{Man}}^{\mathbf{dSpa}} = F_{\mathbf{C}^\infty \mathbf{Sch}}^{\mathbf{dSpa}} \circ F_{\mathbf{Man}}^{\mathbf{C}^\infty \mathbf{Sch}}$.

Write $\hat{\mathbf{C}}^\infty \mathbf{Sch}_{\text{ssc}}^{\text{lf}}$ for the full 2-subcategory of objects \mathbf{X} in \mathbf{dSpa} equivalent to $F_{\mathbf{C}^\infty \mathbf{Sch}}^{\mathbf{dSpa}}(\underline{X})$ for some \underline{X} in $\mathbf{C}^\infty \mathbf{Sch}_{\text{ssc}}^{\text{lf}}$, and $\hat{\mathbf{Man}}$ for the full 2-subcategory of objects \mathbf{X} in \mathbf{dSpa} equivalent to $F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$ for some manifold X . When we say that a d-space \mathbf{X} is a C^∞ -scheme, or is a manifold, we mean that $\mathbf{X} \in \hat{\mathbf{C}}^\infty \mathbf{Sch}_{\text{ssc}}^{\text{lf}}$, or $\mathbf{X} \in \hat{\mathbf{Man}}$, respectively.

In [21, §2.2] we prove:

Theorem 3.2. (a) *Definition 3.1 defines a strict 2-category \mathbf{dSpa} , in which all 2-morphisms are 2-isomorphisms.*

(b) *For any 1-morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ in \mathbf{dSpa} the 2-morphisms $\eta : \mathbf{f} \Rightarrow \mathbf{f}$ form an abelian group under vertical composition, and in fact a real vector space.*

(c) *$F_{\mathbf{C}^\infty \mathbf{Sch}}^{\mathbf{dSpa}}$ and $F_{\mathbf{Man}}^{\mathbf{dSpa}}$ in Definition 3.1 are full and faithful strict 2-functors. Hence $\mathbf{C}^\infty \mathbf{Sch}_{\text{ssc}}^{\text{lf}}$, \mathbf{Man} and $\hat{\mathbf{C}}^\infty \mathbf{Sch}_{\text{ssc}}^{\text{lf}}$, $\hat{\mathbf{Man}}$ are equivalent 2-categories.*

Remark 3.3. One should think of a d-space $\mathbf{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X)$ as being a C^∞ -scheme \underline{X} , which is the ‘classical’ part of \mathbf{X} and lives in a category rather than a 2-category, together with some extra ‘derived’ information $\mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X$. 2-morphisms in \mathbf{dSpa} are wholly to do with this derived part. The sheaf \mathcal{E}_X may be thought of as a (dual) ‘obstruction sheaf’ on \underline{X} .

3.2 Gluing d-spaces by equivalences

Next we discuss gluing of d-spaces and 1-morphisms on open d-subspaces.

Definition 3.4. Let $\mathbf{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X)$ be a d-space. Suppose $\underline{U} \subseteq \underline{X}$ is an open C^∞ -subscheme. Then $\mathbf{U} = (\underline{U}, \mathcal{O}'_X|_{\underline{U}}, \mathcal{E}_X|_{\underline{U}}, \iota_X|_{\underline{U}}, j_X|_{\underline{U}})$ is a d-space. We call \mathbf{U} an *open d-subspace* of \mathbf{X} . An *open cover* of a d-space \mathbf{X} is a family $\{\mathbf{U}_a : a \in A\}$ of open d-subspaces \mathbf{U}_a of \mathbf{X} with $\underline{X} = \bigcup_{a \in A} \underline{U}_a$.

As in [21, §2.4], we can glue 1-morphisms on open d-subspaces which are 2-isomorphic on the overlap. The proof uses partitions of unity, as in §2.2.

Proposition 3.5. Suppose \mathbf{X}, \mathbf{Y} are d-spaces, $\mathbf{U}, \mathbf{V} \subseteq \mathbf{X}$ are open d-subspaces with $\mathbf{X} = \mathbf{U} \cup \mathbf{V}$, $\mathbf{f} : \mathbf{U} \rightarrow \mathbf{Y}$ and $\mathbf{g} : \mathbf{V} \rightarrow \mathbf{Y}$ are 1-morphisms, and $\eta : \mathbf{f}|_{\mathbf{U} \cap \mathbf{V}} \Rightarrow \mathbf{g}|_{\mathbf{U} \cap \mathbf{V}}$ is a 2-morphism. Then there exists a 1-morphism $\mathbf{h} : \mathbf{X} \rightarrow \mathbf{Y}$ and 2-morphisms $\zeta : \mathbf{h}|_{\mathbf{U}} \Rightarrow \mathbf{f}$, $\theta : \mathbf{h}|_{\mathbf{V}} \Rightarrow \mathbf{g}$ such that $\theta|_{\mathbf{U} \cap \mathbf{V}} = \eta \odot \zeta|_{\mathbf{U} \cap \mathbf{V}} : \mathbf{h}|_{\mathbf{U} \cap \mathbf{V}} \Rightarrow \mathbf{g}|_{\mathbf{U} \cap \mathbf{V}}$. This \mathbf{h} is unique up to 2-isomorphism, and independent up to 2-isomorphism of the choice of η .

Equivalences $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ in a 2-category are defined in Appendix A, and are the natural notion of when two objects \mathbf{X}, \mathbf{Y} are ‘the same’. In [21, §2.4] we prove theorems on gluing d-spaces by equivalences. See Spivak [31, Lem. 6.8 & Prop. 6.9] for results similar to Theorem 3.6 for his ‘local C^∞ -ringed spaces’, an ∞ -categorical analogue of our d-spaces.

Theorem 3.6. Suppose \mathbf{X}, \mathbf{Y} are d-spaces, $\mathbf{U} \subseteq \mathbf{X}$, $\mathbf{V} \subseteq \mathbf{Y}$ are open d-subspaces, and $\mathbf{f} : \mathbf{U} \rightarrow \mathbf{V}$ is an equivalence in \mathbf{dSpa} . At the level of topological spaces, we have open $U \subseteq X$, $V \subseteq Y$ with a homeomorphism $f : U \rightarrow V$, so we can form the quotient topological space $Z := X \amalg_f Y = (X \amalg Y) / \sim$, where the equivalence relation \sim on $X \amalg Y$ identifies $u \in U \subseteq X$ with $f(u) \in V \subseteq Y$.

Suppose Z is Hausdorff. Then there exist a d-space \mathbf{Z} with topological space Z , open d-subspaces $\hat{\mathbf{X}}, \hat{\mathbf{Y}}$ in \mathbf{Z} with $\mathbf{Z} = \hat{\mathbf{X}} \cup \hat{\mathbf{Y}}$, equivalences $\mathbf{g} : \mathbf{X} \rightarrow \hat{\mathbf{X}}$ and $\mathbf{h} : \mathbf{Y} \rightarrow \hat{\mathbf{Y}}$ in \mathbf{dSpa} such that $\mathbf{g}|_{\mathbf{U}}$ and $\mathbf{h}|_{\mathbf{V}}$ are both equivalences with $\hat{\mathbf{X}} \cap \hat{\mathbf{Y}}$, and a 2-morphism $\eta : \mathbf{g}|_{\mathbf{U}} \Rightarrow \mathbf{h} \circ \mathbf{f} : \mathbf{U} \rightarrow \hat{\mathbf{X}} \cap \hat{\mathbf{Y}}$. Furthermore, \mathbf{Z} is independent of choices up to equivalence.

Theorem 3.7. Suppose I is an indexing set, and $<$ is a total order on I , and \mathbf{X}_i for $i \in I$ are d-spaces, and for all $i < j$ in I we are given open d-subspaces $\mathbf{U}_{ij} \subseteq \mathbf{X}_i$, $\mathbf{U}_{ji} \subseteq \mathbf{X}_j$ and an equivalence $\mathbf{e}_{ij} : \mathbf{U}_{ij} \rightarrow \mathbf{U}_{ji}$, such that for all $i < j < k$ in I we have a 2-commutative diagram

$$\begin{array}{ccc}
 & \mathbf{U}_{ji} \cap \mathbf{U}_{jk} & \\
 \mathbf{e}_{ij}|_{\mathbf{U}_{ij} \cap \mathbf{U}_{ik}} \nearrow & \Downarrow \eta_{ijk} & \searrow \mathbf{e}_{jk}|_{\mathbf{U}_{ji} \cap \mathbf{U}_{jk}} \\
 \mathbf{U}_{ij} \cap \mathbf{U}_{ik} & \xrightarrow{\mathbf{e}_{ik}|_{\mathbf{U}_{ij} \cap \mathbf{U}_{ik}}} & \mathbf{U}_{ki} \cap \mathbf{U}_{kj}
 \end{array}$$

for some η_{ijk} , where all three 1-morphisms are equivalences.

On the level of topological spaces, define the quotient topological space $Y = (\coprod_{i \in I} X_i) / \sim$, where \sim is the equivalence relation generated by $x_i \sim x_j$ if $i < j$, $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose Y is Hausdorff and second countable. Then there exist a d-space \mathbf{Y} and a 1-morphism $\mathbf{f}_i : \mathbf{X}_i \rightarrow \mathbf{Y}$ which is an equivalence with an open d-subspace $\hat{\mathbf{X}}_i \subseteq \mathbf{Y}$ for all $i \in I$, where $\mathbf{Y} = \bigcup_{i \in I} \hat{\mathbf{X}}_i$, such that $\mathbf{f}_i|_{U_{ij}}$ is an equivalence $U_{ij} \rightarrow \hat{\mathbf{X}}_i \cap \hat{\mathbf{X}}_j$ for all $i < j$ in I , and there exists a 2-morphism $\eta_{ij} : \mathbf{f}_j \circ e_{ij} \Rightarrow \mathbf{f}_i|_{U_{ij}}$. The d-space \mathbf{Y} is unique up to equivalence, and is independent of choice of 2-morphisms η_{ijk} .

Suppose also that \mathbf{Z} is a d-space, and $\mathbf{g}_i : \mathbf{X}_i \rightarrow \mathbf{Z}$ are 1-morphisms for all $i \in I$, and there exist 2-morphisms $\zeta_{ij} : \mathbf{g}_j \circ e_{ij} \Rightarrow \mathbf{g}_i|_{U_{ij}}$ for all $i < j$ in I . Then there exist a 1-morphism $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ and 2-morphisms $\zeta_i : \mathbf{h} \circ \mathbf{f}_i \Rightarrow \mathbf{g}_i$ for all $i \in I$. The 1-morphism \mathbf{h} is unique up to 2-isomorphism, and is independent of the choice of 2-morphisms ζ_{ij} .

Remark 3.8. In Proposition 3.5, it is surprising that \mathbf{h} is independent of η up to 2-isomorphism. It holds because of the existence of *partitions of unity* on nice C^∞ -schemes, as in Proposition 2.8. Here is a sketch proof: suppose $\eta, \mathbf{h}, \zeta, \theta$ and $\eta', \mathbf{h}', \zeta', \theta'$ are alternative choices in Proposition 3.5. Then we have 2-morphisms $(\zeta')^{-1} \odot \zeta : \mathbf{h}|_U \Rightarrow \mathbf{h}'|_U$ and $(\theta')^{-1} \odot \theta : \mathbf{h}|_V \Rightarrow \mathbf{h}'|_V$. Choose a partition of unity $\{\alpha, 1 - \alpha\}$ on \underline{X} subordinate to $\{\underline{U}, \underline{V}\}$, so that $\alpha : \underline{X} \rightarrow \mathbb{R}$ is smooth with α supported on $\underline{U} \subseteq \underline{X}$ and $1 - \alpha$ supported on $\underline{V} \subseteq \underline{X}$. Then $\alpha \cdot ((\zeta')^{-1} \odot \zeta) + (1 - \alpha) \cdot ((\theta')^{-1} \odot \theta)$ is a 2-morphism $\mathbf{h} \Rightarrow \mathbf{h}'$, where $\alpha \cdot ((\zeta')^{-1} \odot \zeta)$ makes sense on all of \underline{X} (rather than just on \underline{U} where $(\zeta')^{-1} \odot \zeta$ is defined) as α is supported on \underline{U} , so we extend by zero on $\underline{X} \setminus \underline{U}$.

Similarly, in Theorem 3.7, the compatibility conditions on the gluing data $\mathbf{X}_i, U_{ij}, e_{ij}$ are significantly weaker than you might expect, because of the existence of partitions of unity. The 2-morphisms η_{ijk} on overlaps $\mathbf{X}_i \cap \mathbf{X}_j \cap \mathbf{X}_k$ are only required to exist, not to satisfy any further conditions. In particular, one might think that on overlaps $\mathbf{X}_i \cap \mathbf{X}_j \cap \mathbf{X}_k \cap \mathbf{X}_l$ we should require

$$\eta_{ikl} \odot (\text{id}_{\mathbf{f}_{kl}} * \eta_{ijk})|_{U_{ij} \cap U_{ik} \cap U_{il}} = \eta_{ijl} \odot (\eta_{jkl} * \text{id}_{\mathbf{f}_{ij}})|_{U_{ij} \cap U_{ik} \cap U_{il}}, \quad (3.3)$$

but we do not. Also, one might expect the ζ_{ij} should satisfy conditions on triple overlaps $\mathbf{X}_i \cap \mathbf{X}_j \cap \mathbf{X}_k$, but they need not.

The moral is that constructing d-spaces by gluing together patches \mathbf{X}_i is straightforward, as one only has to verify mild conditions on triple overlaps $\mathbf{X}_i \cap \mathbf{X}_j \cap \mathbf{X}_k$. Again, this works because of the existence of partitions of unity on nice C^∞ -schemes, which are used to construct the glued d-spaces \mathbf{Z} and 1- and 2-morphisms in Theorems 3.6 and 3.7.

In contrast, for gluing d-stacks in [21, §9.4], we do need compatibility conditions of the form (3.3). The problem of gluing geometric spaces in an ∞ -category \mathcal{C} by equivalences, such as Spivak's derived manifolds [31], is discussed by Toën and Vezzosi [34, §1.3.4] and Lurie [25, §6.1.2]. It requires nontrivial conditions on overlaps $\mathbf{X}_{i_1} \cap \cdots \cap \mathbf{X}_{i_n}$ for all $n = 2, 3, \dots$.

3.3 Fibre products in \mathbf{dSpa}

Fibre products in 2-categories are explained in Appendix A. In [21, §2.5–§2.6] we discuss fibre products in \mathbf{dSpa} , and their relation to transverse fibre products in \mathbf{Man} .

Theorem 3.9. (a) *All fibre products exist in the 2-category \mathbf{dSpa} .*

(b) *Let $g : X \rightarrow Z$ and $h : Y \rightarrow Z$ be smooth maps of manifolds without boundary, and write $\mathbf{X} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$, and similarly for $\mathbf{Y}, \mathbf{Z}, \mathbf{g}, \mathbf{h}$. If g, h are transverse, so that a fibre product $X \times_{g, Z, h} Y$ exists in \mathbf{Man} , then the fibre product $\mathbf{X} \times_{\mathbf{g}, \mathbf{Z}, \mathbf{h}} \mathbf{Y}$ in \mathbf{dSpa} is equivalent in \mathbf{dSpa} to $F_{\mathbf{Man}}^{\mathbf{dSpa}}(X \times_{g, Z, h} Y)$. If g, h are not transverse then $\mathbf{X} \times_{\mathbf{g}, \mathbf{Z}, \mathbf{h}} \mathbf{Y}$ exists in \mathbf{dSpa} , but is not a manifold.*

To prove (a), given 1-morphisms $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$ and $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$, we write down an explicit d-space $\mathbf{W} = (\underline{W}, \mathcal{O}'_W, \mathcal{E}_W, \iota_W, \jmath_W)$, 1-morphisms $\mathbf{e} = (\underline{e}, e', e'') : \mathbf{W} \rightarrow \mathbf{X}$ and $\mathbf{f} = (\underline{f}, f', f'') : \mathbf{W} \rightarrow \mathbf{Y}$ and a 2-morphism $\eta : \mathbf{g} \circ \mathbf{e} \Rightarrow \mathbf{h} \circ \mathbf{f}$, and verify the universal property for

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{\quad} & \mathbf{Y} \\ \downarrow \mathbf{e} & \begin{array}{c} \xrightarrow{f} \nearrow \eta \\ \nwarrow g \end{array} & \downarrow \mathbf{h} \\ \mathbf{X} & \xrightarrow{\quad} & \mathbf{Z} \end{array}$$

to be a 2-Cartesian square in \mathbf{dSpa} . The underlying C^∞ -scheme \underline{W} is the fibre product $\underline{W} = \underline{X} \times_{\underline{g}, \underline{Z}, \underline{h}} \underline{Y}$ in $\mathbf{C}^\infty\mathbf{Sch}$, and $\underline{e} : \underline{W} \rightarrow \underline{X}$, $\underline{f} : \underline{W} \rightarrow \underline{Y}$ are the projections from the fibre product. The definitions of $\mathcal{O}'_W, \iota_W, \jmath_W, e', f'$ are complex, and we will not give them here. The remaining data $\mathcal{E}_W, e'', f'', \eta$, as well as the virtual cotangent sheaf $\phi_W : \mathcal{E}_W \rightarrow \mathcal{F}_W$, is characterized by the following commutative diagram in $\mathbf{qcoh}(\underline{W})$, with exact top row:

$$\begin{array}{ccccc} (g \circ e)^*(\mathcal{E}_Z) & \xrightarrow{\begin{pmatrix} \underline{e}^*(g'') \\ -\underline{f}^*(h'') \\ (g \circ e)^*(\phi_Z) \end{pmatrix}} & \begin{array}{c} \underline{e}^*(\mathcal{E}_X) \oplus \\ \underline{f}^*(\mathcal{E}_Y) \oplus \\ (g \circ e)^*(\mathcal{F}_Z) \end{array} & \xrightarrow{\begin{pmatrix} e'' & f'' & \eta \end{pmatrix}} & \mathcal{E}_W \longrightarrow 0 \\ & \downarrow \begin{pmatrix} -\underline{e}^*(\phi_X) & 0 & \underline{e}^*(g^2) \\ 0 & -\underline{f}^*(\phi_Y) & -\underline{f}^*(h^2) \end{pmatrix} & & \downarrow \phi_W \\ & & \begin{array}{c} \underline{e}^*(\mathcal{F}_X) \oplus \\ \underline{f}^*(\mathcal{F}_Y) \end{array} & \xrightarrow[\cong]{\begin{pmatrix} e^2 & f^2 \end{pmatrix}} & \mathcal{F}_W. \end{array}$$

4 The 2-category of d-manifolds

Sections 4.1–4.8 survey the results of [21, §3–§4] on d-manifolds. Section 4.9 briefly describes extensions to d-manifolds with boundary, d-manifolds with corners, and d-orbifolds from [21, §6–§12], and §4.10 discusses d-manifold bordism and virtual classes for d-manifolds and d-orbifolds following [21, §13]. Section 4.11 explains the relationship between d-manifolds and d-orbifolds and other classes of geometric spaces, summarizing [21, §14].

4.1 The definition of d-manifolds

Definition 4.1. A d-space \mathbf{U} is called a *principal d-manifold* if is equivalent in \mathbf{dSpa} to a fibre product $\mathbf{X} \times_{g, \mathbf{Z}, h} \mathbf{Y}$ with $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \hat{\mathbf{Man}}$. That is,

$$\mathbf{U} \simeq F_{\mathbf{Man}}^{\mathbf{dSpa}}(\mathbf{X}) \times_{F_{\mathbf{Man}}^{\mathbf{dSpa}}(g), F_{\mathbf{Man}}^{\mathbf{dSpa}}(Z), F_{\mathbf{Man}}^{\mathbf{dSpa}}(h)} F_{\mathbf{Man}}^{\mathbf{dSpa}}(\mathbf{Y})$$

for manifolds X, Y, Z and smooth maps $g : X \rightarrow Z$ and $h : Y \rightarrow Z$. The *virtual dimension* $\text{vdim } \mathbf{U}$ of \mathbf{U} is defined to be $\text{vdim } \mathbf{U} = \dim X + \dim Y - \dim Z$. Proposition 4.10(b) below shows that if $\mathbf{U} \neq \emptyset$ then $\text{vdim } \mathbf{U}$ depends only on the d-space \mathbf{U} , and not on the choice of X, Y, Z, g, h , and so is well defined.

A d-space \mathbf{W} is called a *d-manifold of virtual dimension* $n \in \mathbb{Z}$, written $\text{vdim } \mathbf{W} = n$, if \mathbf{W} can be covered by nonempty open d-subspaces \mathbf{U} which are principal d-manifolds with $\text{vdim } \mathbf{U} = n$.

Write \mathbf{dMan} for the full 2-subcategory of d-manifolds in \mathbf{dSpa} . If $\mathbf{X} \in \hat{\mathbf{Man}}$ then $\mathbf{X} \simeq \mathbf{X} \times_* *$, so \mathbf{X} is a principal d-manifold, and thus a d-manifold. Therefore $\hat{\mathbf{Man}}$ is a 2-subcategory of \mathbf{dMan} . We say that a d-manifold \mathbf{X} is a *manifold* if it lies in $\hat{\mathbf{Man}}$. The 2-functor $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \rightarrow \mathbf{dSpa}$ maps into \mathbf{dMan} , and we will write $F_{\mathbf{Man}}^{\mathbf{dMan}} = F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \rightarrow \mathbf{dMan}$.

Here [21, §3.2] are alternative descriptions of principal d-manifolds:

Proposition 4.2. *The following are equivalent characterizations of when a d-space \mathbf{W} is a principal d-manifold:*

- (a) $\mathbf{W} \simeq \mathbf{X} \times_{g, \mathbf{Z}, h} \mathbf{Y}$ for $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \hat{\mathbf{Man}}$.
- (b) $\mathbf{W} \simeq \mathbf{X} \times_{i, \mathbf{Z}, j} \mathbf{Y}$, where X, Y, Z are manifolds, $i : X \rightarrow Z, j : Y \rightarrow Z$ are embeddings, $\mathbf{X} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$, and similarly for Y, Z, i, j . That is, \mathbf{W} is an intersection of two submanifolds X, Y in Z , in the sense of d-spaces.
- (c) $\mathbf{W} \simeq \mathbf{V} \times_{s, \mathbf{E}, 0} \mathbf{V}$, where V is a manifold, $E \rightarrow V$ is a vector bundle, $s : V \rightarrow E$ is a smooth section, $0 : V \rightarrow E$ is the zero section, $\mathbf{V} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V)$, and similarly for $E, s, 0$. That is, \mathbf{W} is the zeroes $s^{-1}(0)$ of a smooth section s of a vector bundle E , in the sense of d-spaces.

4.2 ‘Standard model’ d-manifolds, 1- and 2-morphisms

The next three examples, taken from [21, §3.2 & §3.4], give explicit models for principal d-manifolds in the form $\mathbf{V} \times_{s, \mathbf{E}, 0} \mathbf{V}$ from Proposition 4.2(c) and their 1- and 2-morphisms, which we call *standard models*.

Example 4.3. Let V be a manifold, $E \rightarrow V$ a vector bundle (which we sometimes call the *obstruction bundle*), and $s \in C^\infty(E)$. We will write down an explicit principal d-manifold $\mathbf{S} = (\underline{S}, \mathcal{O}'_S, \mathcal{E}_S, \iota_S, j_S)$ which is equivalent to $\mathbf{V} \times_{s, \mathbf{E}, 0} \mathbf{V}$ in Proposition 4.2(c). We call \mathbf{S} the *standard model* of (V, E, s) , and also write it $\mathbf{S}_{V, E, s}$. Proposition 4.2 shows that every principal d-manifold \mathbf{W} is equivalent to $\mathbf{S}_{V, E, s}$ for some V, E, s .

Write $C^\infty(V)$ for the C^∞ -ring of smooth functions $c : V \rightarrow \mathbb{R}$, and $C^\infty(E)$, $C^\infty(E^*)$ for the vector spaces of smooth sections of E, E^* over V . Then s lies in $C^\infty(E)$, and $C^\infty(E), C^\infty(E^*)$ are modules over $C^\infty(V)$, and there is a natural bilinear product $\cdot : C^\infty(E^*) \times C^\infty(E) \rightarrow C^\infty(V)$. Define $I_s \subseteq C^\infty(V)$ to be the ideal generated by s . That is,

$$I_s = \{\alpha \cdot s : \alpha \in C^\infty(E^*)\} \subseteq C^\infty(V). \quad (4.1)$$

Let $I_s^2 = \langle fg : f, g \in I_s \rangle_{\mathbb{R}}$ be the square of I_s . Then I_s^2 is an ideal in $C^\infty(V)$, the ideal generated by $s \otimes s \in C^\infty(E \otimes E)$. That is,

$$I_s^2 = \{\beta \cdot (s \otimes s) : \beta \in C^\infty(E^* \otimes E^*)\} \subseteq C^\infty(V).$$

Define C^∞ -rings $\mathfrak{C} = C^\infty(V)/I_s$, $\mathfrak{C}' = C^\infty(V)/I_s^2$, and let $\pi : \mathfrak{C}' \rightarrow \mathfrak{C}$ be the natural projection from the inclusion $I_s^2 \subseteq I_s$. Define a topological space $S = \{v \in V : s(v) = 0\}$, as a subspace of V . Now $s(v) = 0$ if and only if $(s \otimes s)(v) = 0$. Thus S is the underlying topological space for both $\text{Spec } \mathfrak{C}$ and $\text{Spec } \mathfrak{C}'$. So $\text{Spec } \mathfrak{C} = \underline{S} = (S, \mathcal{O}_S)$, $\text{Spec } \mathfrak{C}' = \underline{S}' = (S, \mathcal{O}'_S)$, and $\text{Spec } \pi = \iota_S = (\text{id}_S, \iota_S) : \underline{S}' \rightarrow \underline{S}$, where $\underline{S}, \underline{S}'$ are fair affine C^∞ -schemes, and $\mathcal{O}_S, \mathcal{O}'_S$ are sheaves of C^∞ -rings on S , and $\iota_S : \mathcal{O}'_S \rightarrow \mathcal{O}_S$ is a morphism of sheaves of C^∞ -rings. Since π is surjective with kernel the square zero ideal I_s/I_s^2 , ι_S is surjective, with kernel \mathcal{I}_S a sheaf of square zero ideals in \mathcal{O}'_S .

From (4.1) we have a surjective $C^\infty(V)$ -module morphism $C^\infty(E^*) \rightarrow I_s$ mapping $\alpha \mapsto \alpha \cdot s$. Applying $\otimes_{C^\infty(V)} \mathfrak{C}$ gives a surjective \mathfrak{C} -module morphism

$$\sigma : C^\infty(E^*)/(I_s \cdot C^\infty(E^*)) \longrightarrow I_s/I_s^2, \quad \sigma : \alpha + (I_s \cdot C^\infty(E^*)) \longmapsto \alpha \cdot s + I_s^2.$$

Define $\mathcal{E}_S = \text{MSpec}(C^\infty(E^*)/(I_s \cdot C^\infty(E^*)))$. Also $\text{MSpec}(I_s/I_s^2) = \mathcal{I}_S$, so $j_S = \text{MSpec } \sigma$ is a surjective morphism $j_S : \mathcal{E}_S \rightarrow \mathcal{I}_S$ in $\text{qcoh}(\underline{S})$. Therefore $\mathbf{S}_{V,E,s} = \mathbf{S} = (\underline{S}, \mathcal{O}'_S, \mathcal{E}_S, \iota_S, j_S)$ is a d-space.

In fact \mathcal{E}_S is a vector bundle on \underline{S} naturally isomorphic to $\underline{E}^*|_{\underline{S}}$, where \underline{E} is the vector bundle on $\underline{V} = F_{\text{Man}}^{\text{C}^\infty \text{Sch}}(V)$ corresponding to $E \rightarrow V$. Also $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$. The morphism $\phi_S : \mathcal{E}_S \rightarrow \mathcal{F}_S$ can be interpreted as follows: choose a connection ∇ on $E \rightarrow V$. Then $\nabla s \in C^\infty(E \otimes T^*V)$, so we can regard ∇s as a morphism of vector bundles $E^* \rightarrow T^*V$ on V . This lifts to a morphism of vector bundles $\hat{\nabla}s : \underline{E}^* \rightarrow T^*\underline{V}$ on the C^∞ -scheme \underline{V} , and ϕ_S is identified with $\hat{\nabla}s|_{\underline{S}} : \underline{E}^*|_{\underline{S}} \rightarrow T^*\underline{V}|_{\underline{S}}$ under the isomorphisms $\mathcal{E}_S \cong \underline{E}^*|_{\underline{S}}$, $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$.

Proposition 4.2 implies that every principal d-manifold \mathbf{W} is equivalent to $\mathbf{S}_{V,E,s}$ for some V, E, s . The notation $O(s)$ and $O(s^2)$ used below should be interpreted as follows. Let V be a manifold, $E \rightarrow V$ a vector bundle, and $s \in C^\infty(E)$. If $F \rightarrow V$ is another vector bundle and $t \in C^\infty(F)$, then we write $t = O(s)$ if $t = \alpha \cdot s$ for some $\alpha \in C^\infty(F \otimes E^*)$, and $t = O(s^2)$ if $t = \beta \cdot (s \otimes s)$ for some $\beta \in C^\infty(F \otimes E^* \otimes E^*)$. Similarly, if W is a manifold and $f, g : V \rightarrow W$ are smooth then we write $f = g + O(s)$ if $c \circ f - c \circ g = O(s)$ for all smooth $c : W \rightarrow \mathbb{R}$, and $f = g + O(s^2)$ if $c \circ f - c \circ g = O(s^2)$ for all c .

Example 4.4. Let V, W be manifolds, $E \rightarrow V$, $F \rightarrow W$ be vector bundles, and $s \in C^\infty(E)$, $t \in C^\infty(F)$. Write $\mathbf{X} = \mathbf{S}_{V,E,s}$, $\mathbf{Y} = \mathbf{S}_{W,F,t}$ for the ‘standard model’ principal d-manifolds from Example 4.3. Suppose $f : V \rightarrow W$ is a smooth map, and $\hat{f} : E \rightarrow f^*(F)$ is a morphism of vector bundles on V satisfying

$$\hat{f} \circ s = f^*(t) + O(s^2) \quad \text{in } C^\infty(f^*(F)). \quad (4.2)$$

We will define a 1-morphism $\mathbf{g} = (g, g', g'') : \mathbf{X} \rightarrow \mathbf{Y}$ in \mathbf{dMan} using f, \hat{f} . We will also write $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$ as $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$, and call it a *standard model 1-morphism*. If $x \in X$ then $x \in V$ with $s(x) = 0$, so (4.2) implies that

$$t(f(x)) = (f^*(t))(x) = \hat{f}(s(x)) + O(s(x)^2) = 0,$$

so $f(x) \in Y \subseteq W$. Thus $g := f|_X$ maps $X \rightarrow Y$.

Define morphisms of C^∞ -rings

$$\begin{aligned} \phi : C^\infty(W)/I_t &\longrightarrow C^\infty(V)/I_s, & \phi' : C^\infty(W)/I_t^2 &\longrightarrow C^\infty(V)/I_s^2, \\ \text{by } \phi : c + I_t &\longmapsto c \circ f + I_s, & \phi' : c + I_t^2 &\longmapsto c \circ f + I_s^2. \end{aligned}$$

Here ϕ is well-defined since if $c \in I_t$ then $c = \gamma \cdot t$ for some $\gamma \in C^\infty(F^*)$, so

$$c \circ f = (\gamma \cdot t) \circ f = f^*(\gamma) \cdot f^*(t) = f^*(\gamma) \cdot (\hat{f} \circ s + O(s^2)) = (\hat{f} \circ f^*(\gamma)) \cdot s + O(s^2) \in I_s.$$

Similarly if $c \in I_t^2$ then $c \circ f \in I_s^2$, so ϕ' is well-defined. Thus we have C^∞ -scheme morphisms $\underline{g} = (g, g^\sharp) = \text{Spec } \phi : \underline{X} \rightarrow \underline{Y}$, and $(g, g') = \text{Spec } \phi' : (X, \mathcal{O}'_X) \rightarrow (Y, \mathcal{O}'_Y)$, both with underlying map g . Hence $g^\sharp : g^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$ and $g' : g^{-1}(\mathcal{O}'_Y) \rightarrow \mathcal{O}'_X$ are morphisms of sheaves of C^∞ -rings on X .

Since $\underline{g}^*(\mathcal{E}_Y) = \text{MSpec}(C^\infty(f^*(F^*))/ (I_s \cdot C^\infty(f^*(F^*))))$, we may define $g'' : \underline{g}^*(\mathcal{E}_Y) \rightarrow \mathcal{E}_X$ by $g'' = \text{MSpec}(G'')$, where

$$\begin{aligned} G'' : C^\infty(f^*(F^*))/ (I_s \cdot C^\infty(f^*(F^*))) &\longrightarrow C^\infty(E^*)/ (I_s \cdot C^\infty(E^*)) \\ \text{is defined by } G'' : \gamma + I_s \cdot C^\infty(f^*(F^*)) &\longmapsto \gamma \circ \hat{f} + I_s \cdot C^\infty(E^*). \end{aligned}$$

This defines $\mathbf{g} = (g, g', g'')$. One can show it is a 1-morphism $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$ in \mathbf{dSpa} , which we also write as $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$.

Now suppose \tilde{V} is an open neighbourhood of $s^{-1}(0)$ in V , and let $\tilde{E} = E|_{\tilde{V}}$ and $\tilde{s} = s|_{\tilde{V}}$. Write $i_{\tilde{V}} : \tilde{V} \rightarrow V$ for the inclusion. Then $i_{\tilde{V}}^*(E) = \tilde{E}$, and $\text{id}_{\tilde{E}} \circ \tilde{s} = \tilde{s} = i_{\tilde{V}}^*(s)$. Thus we have a 1-morphism $i_{\tilde{V},V} = \mathbf{S}_{i_{\tilde{V}},\text{id}_{\tilde{E}}} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \rightarrow \mathbf{S}_{V,E,s}$. It is easy to show that $i_{\tilde{V},V}$ is a 1-*isomorphism*, with an inverse $i_{\tilde{V},V}^{-1}$. That is, making V smaller without making $s^{-1}(0)$ smaller does not really change $\mathbf{S}_{V,E,s}$; the d-manifold $\mathbf{S}_{V,E,s}$ depends only on E, s on an arbitrarily small open neighbourhood of $s^{-1}(0)$ in V .

Example 4.5. Let V, W be manifolds, $E \rightarrow V$, $F \rightarrow W$ be vector bundles, and $s \in C^\infty(E)$, $t \in C^\infty(F)$. Suppose $f, g : V \rightarrow W$ are smooth and $\hat{f} : E \rightarrow f^*(F)$, $\hat{g} : E \rightarrow g^*(F)$ are vector bundle morphisms with $\hat{f} \circ s = f^*(t) + O(s^2)$ and

$\hat{g} \circ s = g^*(t) + O(s^2)$, so we have 1-morphisms $\mathbf{S}_{f,\hat{f}}, \mathbf{S}_{g,\hat{g}} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$. It is easy to show that $\mathbf{S}_{f,\hat{f}} = \mathbf{S}_{g,\hat{g}}$ if and only if $g = f + O(s^2)$ and $\hat{g} = \hat{f} + O(s)$.

Now suppose $\Lambda : E \rightarrow f^*(TW)$ is a morphism of vector bundles on V . Taking the dual of Λ and lifting to \underline{V} gives $\Lambda^* : f^*(T^*\underline{W}) \rightarrow \mathcal{E}^*$. Restricting to the C^∞ -subscheme $\underline{X} = s^{-1}(0)$ in \underline{V} gives $\lambda = \Lambda^*|_{\underline{X}} : f^*(\mathcal{F}_Y) \cong f^*(T^*\underline{W})|_{\underline{X}} \rightarrow \mathcal{E}^*|_{\underline{X}} = \mathcal{E}_X$. One can show that λ is a 2-morphism $\mathbf{S}_{f,\hat{f}} \Rightarrow \mathbf{S}_{g,\hat{g}}$ if and only if

$$g = f + \Lambda \circ s + O(s^2) \quad \text{and} \quad \hat{g} = \hat{f} + f^*(dt) \circ \Lambda + O(s).$$

We write λ as $\mathbf{S}_\Lambda : \mathbf{S}_{f,\hat{f}} \Rightarrow \mathbf{S}_{g,\hat{g}}$, and call it a *standard model 2-morphism*. Every 2-morphism $\eta : \mathbf{S}_{f,\hat{f}} \Rightarrow \mathbf{S}_{g,\hat{g}}$ is \mathbf{S}_Λ for some Λ . Two vector bundle morphisms $\Lambda, \Lambda' : E \rightarrow f^*(TW)$ have $\mathbf{S}_\Lambda = \mathbf{S}_{\Lambda'}$ if and only if $\Lambda = \Lambda' + O(s)$.

If \mathbf{X} is a d-manifold and $x \in \mathbf{X}$ then x has an open neighbourhood \mathbf{U} in \mathbf{X} equivalent in \mathbf{dSpa} to $\mathbf{S}_{V,E,s}$ for some manifold V , vector bundle $E \rightarrow V$ and $s \in C^\infty(E)$. In [21, §3.3] we investigate the extent to which \mathbf{X} determines V, E, s near a point in \mathbf{X} and V , and prove:

Theorem 4.6. *Let \mathbf{X} be a d-manifold, and $x \in \mathbf{X}$. Then there exists an open neighbourhood \mathbf{U} of x in \mathbf{X} and an equivalence $\mathbf{U} \simeq \mathbf{S}_{V,E,s}$ in \mathbf{dMan} for some manifold V , vector bundle $E \rightarrow V$ and $s \in C^\infty(E)$ which identifies $x \in \mathbf{U}$ with a point $v \in V$ such that $s(v) = ds(v) = 0$, where $\mathbf{S}_{V,E,s}$ is as in Example 4.3. These V, E, s are determined up to non-canonical isomorphism near v by \mathbf{X} near x , and in fact they depend only on the underlying C^∞ -scheme \underline{X} and the integer $\text{vdim } \mathbf{X}$.*

Thus, if we impose the extra condition $ds(v) = 0$, which is in fact equivalent to choosing V, E, s with $\dim V$ as small as possible, then V, E, s are determined uniquely near v by \mathbf{X} near x (that is, V, E, s are determined locally up to isomorphism, but not up to canonical isomorphism). If we drop the condition $ds(v) = 0$ then V, E, s are determined uniquely near v by \mathbf{X} near x and $\dim V$.

Theorem 4.6 shows that any d-manifold $\mathbf{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X)$ is determined up to equivalence in \mathbf{dSpa} near any point $x \in \mathbf{X}$ by the ‘classical’ underlying C^∞ -scheme \underline{X} and the integer $\text{vdim } \mathbf{X}$. So we can ask: what extra information about \mathbf{X} is contained in the ‘derived’ data $\mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X$? One can think of this extra information as like a vector bundle \mathcal{E} over \underline{X} . The only local information in a vector bundle \mathcal{E} is $\text{rank } \mathcal{E} \in \mathbb{Z}$, but globally it also contains nontrivial algebraic-topological information.

Suppose now that $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a 1-morphism in \mathbf{dMan} , and $x \in \mathbf{X}$ with $\mathbf{f}(x) = y \in \mathbf{Y}$. Then by Theorem 4.6 we have $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ near x and $\mathbf{Y} \simeq \mathbf{S}_{W,F,t}$ near y . So up to composition with equivalences, we can identify \mathbf{f} near x with a 1-morphism $\mathbf{g} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$. Thus, to understand arbitrary 1-morphisms \mathbf{f} in \mathbf{dMan} near a point, it is enough to study 1-morphisms $\mathbf{g} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$. Our next theorem, proved in [21, §3.4], shows that after making V smaller, every 1-morphism $\mathbf{g} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$ is of the form $\mathbf{S}_{f,\hat{f}}$.

Theorem 4.7. *Let V, W be manifolds, $E \rightarrow V, F \rightarrow W$ be vector bundles, and $s \in C^\infty(E), t \in C^\infty(F)$. Define principal d-manifolds $\mathbf{X} = \mathbf{S}_{V,E,s}, \mathbf{Y} = \mathbf{S}_{W,F,t}$,*

with topological spaces $X = \{v \in V : s(v) = 0\}$ and $Y = \{w \in W : t(w) = 0\}$. Suppose $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$ is a 1-morphism. Then there exist an open neighbourhood \tilde{V} of X in V , a smooth map $f : \tilde{V} \rightarrow W$, and a morphism of vector bundles $\hat{f} : \tilde{E} \rightarrow f^*(F)$ with $\hat{f} \circ \tilde{s} = f^*(t)$, where $\tilde{E} = E|_{\tilde{V}}$, $\tilde{s} = s|_{\tilde{V}}$, such that $\mathbf{g} = \mathbf{S}_{f,\hat{f}} \circ \mathbf{i}_{\tilde{V},V}^{-1}$, where $\mathbf{i}_{\tilde{V},V} = \mathbf{S}_{\text{id}_{\tilde{V}},\text{id}_{\tilde{E}}} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \rightarrow \mathbf{S}_{V,E,s}$ is a 1-isomorphism, and $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \rightarrow \mathbf{S}_{W,F,t}$.

These results give a good differential-geometric picture of d-manifolds and their 1- and 2-morphisms near a point. The $O(s)$ and $O(s^2)$ notation helps keep track of what information from V, E, s and f, \hat{f} and Λ is remembered and what forgotten by the d-manifolds $\mathbf{S}_{V,E,s}$, 1-morphisms $\mathbf{S}_{f,\hat{f}}$ and 2-morphisms \mathbf{S}_{Λ} .

4.3 The 2-category of virtual vector bundles

In our theory of derived differential geometry, it is a general principle that categories in classical differential geometry should often be replaced by 2-categories, and classical concepts be replaced by 2-categorical analogues.

In classical differential geometry, if X is a manifold, the vector bundles $E \rightarrow X$ and their morphisms form a category $\text{vect}(X)$. The cotangent bundle T^*X is an important example of a vector bundle. If $f : X \rightarrow Y$ is smooth then pullback $f^* : \text{vect}(Y) \rightarrow \text{vect}(X)$ is a functor. There is a natural morphism $\text{df}^* : f^*(T^*Y) \rightarrow T^*X$. We now explain 2-categorical analogues of all this for d-manifolds, following [21, §3.1–§3.2].

Definition 4.8. Let \underline{X} be a C^∞ -scheme, which will usually be the C^∞ -scheme underlying a d-manifold \mathbf{X} . We will define a 2-category $\text{vqcoh}(\underline{X})$ of *virtual quasicohherent sheaves* on \underline{X} . Objects of $\text{vqcoh}(\underline{X})$ are morphisms $\phi : \mathcal{E}^1 \rightarrow \mathcal{E}^2$ in $\text{qcoh}(\underline{X})$, which we also may write as $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ or $(\mathcal{E}^\bullet, \phi)$. Given objects $\phi : \mathcal{E}^1 \rightarrow \mathcal{E}^2$ and $\psi : \mathcal{F}^1 \rightarrow \mathcal{F}^2$, a 1-morphism $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ is a pair of morphisms $f^1 : \mathcal{E}^1 \rightarrow \mathcal{F}^1$, $f^2 : \mathcal{E}^2 \rightarrow \mathcal{F}^2$ in $\text{qcoh}(\underline{X})$ such that $\psi \circ f^1 = f^2 \circ \phi$. We write f^\bullet for (f^1, f^2) .

The *identity* 1-morphism of $(\mathcal{E}^\bullet, \phi)$ is $(\text{id}_{\mathcal{E}^1}, \text{id}_{\mathcal{E}^2})$. The *composition* of 1-morphisms $f^\bullet : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ and $g^\bullet : (\mathcal{F}^\bullet, \psi) \rightarrow (\mathcal{G}^\bullet, \xi)$ is $g^\bullet \circ f^\bullet = (g^1 \circ f^1, g^2 \circ f^2) : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{G}^\bullet, \xi)$.

Given $f^\bullet, g^\bullet : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$, a 2-morphism $\eta : f^\bullet \Rightarrow g^\bullet$ is a morphism $\eta : \mathcal{E}^2 \rightarrow \mathcal{F}^1$ in $\text{qcoh}(\underline{X})$ such that $g^1 = f^1 + \eta \circ \phi$ and $g^2 = f^2 + \psi \circ \eta$. The *identity* 2-morphism for f^\bullet is $\text{id}_{f^\bullet} = 0$. If $f^\bullet, g^\bullet, h^\bullet : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ are 1-morphisms and $\eta : f^\bullet \Rightarrow g^\bullet$, $\zeta : g^\bullet \Rightarrow h^\bullet$ are 2-morphisms, the *vertical composition* of 2-morphisms $\zeta \odot \eta : f^\bullet \Rightarrow h^\bullet$ is $\zeta \odot \eta = \zeta + \eta$. If $f^\bullet, \tilde{f}^\bullet : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ and $g^\bullet, \tilde{g}^\bullet : (\mathcal{F}^\bullet, \psi) \rightarrow (\mathcal{G}^\bullet, \xi)$ are 1-morphisms and $\eta : f^\bullet \Rightarrow \tilde{f}^\bullet$, $\zeta : g^\bullet \Rightarrow \tilde{g}^\bullet$ are 2-morphisms, the *horizontal composition* of 2-morphisms $\zeta * \eta : g^\bullet \circ f^\bullet \Rightarrow \tilde{g}^\bullet \circ \tilde{f}^\bullet$ is $\zeta * \eta = g^1 \circ \eta + \zeta \circ f^2 + \zeta \circ \psi \circ \eta$. This defines a strict 2-category $\text{vqcoh}(\underline{X})$, the obvious 2-category of 2-term complexes in $\text{qcoh}(\underline{X})$.

If $\underline{U} \subseteq \underline{X}$ is an open C^∞ -subscheme then restriction from \underline{X} to \underline{U} defines a strict 2-functor $|_{\underline{U}} : \text{vqcoh}(\underline{X}) \rightarrow \text{vqcoh}(\underline{U})$. An object $(\mathcal{E}^\bullet, \phi)$ in $\text{vqcoh}(\underline{X})$ is called a *virtual vector bundle of rank $d \in \mathbb{Z}$* if \underline{X} may be covered by open $\underline{U} \subseteq \underline{X}$ such that $(\mathcal{E}^\bullet, \phi)|_{\underline{U}}$ is equivalent in $\text{vqcoh}(\underline{U})$ to some $(\mathcal{F}^\bullet, \psi)$ for $\mathcal{F}^1, \mathcal{F}^2$ vector

bundles on \underline{U} with $\text{rank } \mathcal{F}^2 - \text{rank } \mathcal{F}^1 = d$. We write $\text{rank}(\mathcal{E}^\bullet, \phi) = d$. If $\underline{X} \neq \emptyset$ then $\text{rank}(\mathcal{E}^\bullet, \phi)$ depends only on $\mathcal{E}^1, \mathcal{E}^2, \phi$, so it is well-defined. Write $\text{vvect}(\underline{X})$ for the full 2-subcategory of virtual vector bundles in $\text{vqcoh}(\underline{X})$.

If $\underline{f} : \underline{X} \rightarrow \underline{Y}$ is a C^∞ -scheme morphism then pullback gives a strict 2-functor $\underline{f}^* : \text{vqcoh}(\underline{Y}) \rightarrow \text{vqcoh}(\underline{X})$, which maps $\text{vvect}(\underline{Y}) \rightarrow \text{vvect}(\underline{X})$.

We apply these ideas to d-spaces.

Definition 4.9. Let $\mathbf{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X)$ be a d-space. Define the *virtual cotangent sheaf* $T^*\mathbf{X}$ of \mathbf{X} to be the morphism $\phi_X : \mathcal{E}_X \rightarrow \mathcal{F}_X$ in $\text{qcoh}(\underline{X})$ from Definition 3.1, regarded as a virtual quasicoherent sheaf on \underline{X} .

Let $\mathbf{f} = (\underline{f}, f', f'') : \mathbf{X} \rightarrow \mathbf{Y}$ be a 1-morphism in \mathbf{dSpa} . Then $T^*\mathbf{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ and $\underline{f}^*(T^*\mathbf{Y}) = (\underline{f}^*(\mathcal{E}_Y), \underline{f}^*(\mathcal{F}_Y), \underline{f}^*(\phi_Y))$ are virtual quasicoherent sheaves on \underline{X} , and $\Omega_{\mathbf{f}} := (f'', f^2)$ is a 1-morphism $\underline{f}^*(T^*\mathbf{Y}) \rightarrow T^*\mathbf{X}$ in $\text{vqcoh}(\underline{X})$, as (3.1) commutes.

Let $\mathbf{f}, \mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$ be 1-morphisms in \mathbf{dSpa} , and $\eta : \mathbf{f} \Rightarrow \mathbf{g}$ a 2-morphism. Then $\eta : \underline{f}^*(\mathcal{F}_Y) \rightarrow \mathcal{E}_X$ with $g'' = f'' + \eta \circ \underline{f}^*(\phi_Y)$ and $g^2 = f^2 + \phi_X \circ \eta$, as in (3.2). It follows that η is a 2-morphism $\Omega_{\mathbf{f}} \Rightarrow \Omega_{\mathbf{g}}$ in $\text{vqcoh}(\underline{X})$. Thus, objects, 1-morphisms and 2-morphisms in \mathbf{dSpa} lift to objects, 1-morphisms and 2-morphisms in $\text{vqcoh}(\underline{X})$.

The next proposition justifies the definition of virtual vector bundle. Because of part (b), if \mathbf{W} is a d-manifold we call $T^*\mathbf{W}$ the *virtual cotangent bundle* of \mathbf{W} , rather than the virtual cotangent sheaf.

Proposition 4.10. (a) Let V be a manifold, $E \rightarrow V$ a vector bundle, and $s \in C^\infty(E)$. Then Example 4.3 defines a principal d-manifold $\mathbf{S}_{V,E,s}$. Its cotangent bundle $T^*\mathbf{S}_{V,E,s}$ is a virtual vector bundle on $\underline{\mathbf{S}_{V,E,s}}$ of rank $\dim V - \text{rank } E$.

(b) Let \mathbf{W} be a d-manifold. Then $T^*\mathbf{W}$ is a virtual vector bundle on $\underline{\mathbf{W}}$ of rank $\text{vdim } \mathbf{W}$. Hence if $\mathbf{W} \neq \emptyset$ then $\text{vdim } \mathbf{W}$ is well-defined.

The virtual cotangent bundle $T^*\mathbf{X}$ of a d-manifold \mathbf{X} contains only a fraction of the information in $\mathbf{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \iota_X, j_X)$, but many interesting properties of d-manifolds \mathbf{X} and 1-morphisms $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ can be expressed solely in terms of virtual cotangent bundles $T^*\mathbf{X}, T^*\mathbf{Y}$ and 1-morphisms $\Omega_{\mathbf{f}} : \underline{f}^*(T^*\mathbf{Y}) \rightarrow T^*\mathbf{X}$. Here is an example of this.

Definition 4.11. Let \underline{X} be a C^∞ -scheme. We say that a virtual vector bundle $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ on \underline{X} is a *vector bundle* if it is equivalent in $\text{vvect}(\underline{X})$ to $(0, \mathcal{E}, 0)$ for some vector bundle \mathcal{E} on \underline{X} . One can show $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ is a vector bundle if and only if ϕ has a left inverse in $\text{qcoh}(\underline{X})$.

Proposition 4.12. Let \mathbf{X} be a d-manifold. Then \mathbf{X} is a manifold (that is, $\mathbf{X} \in \hat{\mathbf{Man}}$) if and only if $T^*\mathbf{X}$ is a vector bundle, or equivalently, if $\phi_X : \mathcal{E}_X \rightarrow \mathcal{F}_X$ has a left inverse in $\text{qcoh}(\underline{X})$.

4.4 Equivalences in \mathbf{dMan} , and gluing by equivalences

Equivalences in a 2-category are defined in Appendix A. Equivalences in \mathbf{dMan} are the best derived analogue of isomorphisms in \mathbf{Man} , that is, of diffeomorphisms of manifolds. A smooth map of manifolds $f : X \rightarrow Y$ is called *étale* if it is a local diffeomorphism. Here is the derived analogue.

Definition 4.13. Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a 1-morphism in \mathbf{dMan} . We call \mathbf{f} *étale* if it is a *local equivalence*, that is, if for each $x \in \mathbf{X}$ there exist open $x \in U \subseteq \mathbf{X}$ and $\mathbf{f}(x) \in V \subseteq \mathbf{Y}$ such that $\mathbf{f}(U) = V$ and $\mathbf{f}|_U : U \rightarrow V$ is an equivalence.

If $f : X \rightarrow Y$ is a smooth map of manifolds, then f is étale if and only if $df^* : f^*(T^*Y) \rightarrow T^*X$ is an isomorphism of vector bundles. (The analogue is false for schemes.) In [21, §3.5] we prove a version of this for d-manifolds:

Theorem 4.14. Suppose $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a 1-morphism of d-manifolds. Then the following are equivalent:

- (i) \mathbf{f} is étale;
- (ii) $\Omega_{\mathbf{f}} : \underline{f}^*(T^*\mathbf{Y}) \rightarrow T^*\mathbf{X}$ is an equivalence in $\mathbf{vqcoh}(\underline{X})$; and
- (iii) the following is a split short exact sequence in $\mathbf{qcoh}(\underline{X})$:

$$0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \xrightarrow{f'' \oplus -\underline{f}^*(\phi_Y)} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \xrightarrow{\phi_X \oplus f^2} \mathcal{F}_X \longrightarrow 0.$$

If in addition $f : X \rightarrow Y$ is a bijection, then \mathbf{f} is an equivalence in \mathbf{dMan} .

The analogue of Theorem 4.14 for d-spaces is false. When $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a ‘standard model’ 1-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$, as in §4.2, we can express the conditions for $\mathbf{S}_{f,\hat{f}}$ to be étale or an equivalence in terms of f, \hat{f} .

Theorem 4.15. Let V, W be manifolds, $E \rightarrow V, F \rightarrow W$ be vector bundles, $s \in C^\infty(E), t \in C^\infty(F), f : V \rightarrow W$ be smooth, and $\hat{f} : E \rightarrow f^*(F)$ be a morphism of vector bundles on V with $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Example 4.4 defines a 1-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{W,F,t}$ in \mathbf{dMan} . This $\mathbf{S}_{f,\hat{f}}$ is étale if and only if for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, the following sequence of vector spaces is exact:

$$0 \longrightarrow T_v V \xrightarrow{ds(v) \oplus df(v)} E_v \oplus T_w W \xrightarrow{\hat{f}(v) \oplus -dt(w)} F_w \longrightarrow 0.$$

Also $\mathbf{S}_{f,\hat{f}}$ is an equivalence if and only if in addition $f|_{s^{-1}(0)} : s^{-1}(0) \rightarrow t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{v \in V : s(v) = 0\}, t^{-1}(0) = \{w \in W : t(w) = 0\}$.

Section 3.2 discussed gluing d-spaces by equivalences on open d-subspaces. It generalizes immediately to d-manifolds: if in Theorem 3.7 we fix $n \in \mathbb{Z}$ and take the initial d-spaces \mathbf{X}_i to be d-manifolds with $\text{vdim } \mathbf{X}_i = n$, then the glued d-space \mathbf{Y} is also a d-manifold with $\text{vdim } \mathbf{Y} = n$.

Here is an analogue of Theorem 3.7, taken from [21, §3.6], in which we take the d-spaces \mathbf{X}_i to be ‘standard model’ d-manifolds $\mathbf{S}_{V_i, E_i, s_i}$, and the 1-morphisms \mathbf{e}_{ij} to be ‘standard model’ 1-morphisms $\mathbf{S}_{e_{ij}, \hat{e}_{ij}}$. We also use Theorem 4.15 in (iii) to characterize when \mathbf{e}_{ij} is an equivalence.

Theorem 4.16. *Suppose we are given the following data:*

- (a) *an integer n ;*
- (b) *a Hausdorff, second countable topological space X ;*
- (c) *an indexing set I , and a total order $<$ on I ;*
- (d) *for each i in I , a manifold V_i , a vector bundle $E_i \rightarrow V_i$ with $\dim V_i - \text{rank } E_i = n$, a smooth section $s_i : V_i \rightarrow E_i$, and a homeomorphism $\psi_i : X_i \rightarrow \hat{X}_i$, where $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ and $\hat{X}_i \subseteq X$ is open; and*
- (e) *for all $i < j$ in I , an open submanifold $V_{ij} \subseteq V_i$, a smooth map $e_{ij} : V_{ij} \rightarrow V_j$, and a morphism of vector bundles $\hat{e}_{ij} : E_i|_{V_{ij}} \rightarrow e_{ij}^*(E_j)$.*

Using notation $O(s_i), O(s_i^2)$ as in §4.2, let this data satisfy the conditions:

- (i) $X = \bigcup_{i \in I} \hat{X}_i$;
- (ii) *if $i < j$ in I then $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j) + O(s_i^2)$, $\psi_i(X_i \cap V_{ij}) = \hat{X}_i \cap \hat{X}_j$, and $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, and if $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i)$ then the following is exact:*

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{ds_i(v_i) \oplus de_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -ds_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

- (iii) *if $i < j < k$ in I then*

$$\begin{aligned} e_{ik}|_{V_{ij} \cap V_{ik}} &= e_{jk} \circ e_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i^2) \quad \text{and} \\ \hat{e}_{ik}|_{V_{ij} \cap V_{ik}} &= e_{ij}|_{V_{ij} \cap V_{ik}}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i). \end{aligned}$$

Then there exist a d -manifold \mathbf{X} with $\text{vdim } \mathbf{X} = n$ and underlying topological space X , and a 1-morphism $\psi_i : \mathbf{S}_{V_i, E_i, s_i} \rightarrow \mathbf{X}$ with underlying continuous map ψ_i which is an equivalence with the open d -submanifold $\hat{X}_i \subseteq \mathbf{X}$ corresponding to $\hat{X}_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\eta_{ij} : \psi_j \circ \mathbf{S}_{e_{ij}, \hat{e}_{ij}} \Rightarrow \psi_i \circ \mathbf{i}_{V_{ij}, V_i}$, where $\mathbf{S}_{e_{ij}, \hat{e}_{ij}} : \mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}} \rightarrow \mathbf{S}_{V_j, E_j, s_j}$ and $\mathbf{i}_{V_{ij}, V_i} : \mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}} \rightarrow \mathbf{S}_{V_i, E_i, s_i}$. This d -manifold \mathbf{X} is unique up to equivalence in \mathbf{dMan} .

Suppose also that Y is a manifold, and $g_i : V_i \rightarrow Y$ are smooth maps for all $i \in I$, and $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i)$ for all $i < j$ in I . Then there exist a 1-morphism $\mathbf{h} : \mathbf{X} \rightarrow \mathbf{Y}$ unique up to 2-isomorphism, where $\mathbf{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y) = \mathbf{S}_{Y, 0, 0}$, and 2-morphisms $\zeta_i : \mathbf{h} \circ \psi_i \Rightarrow \mathbf{S}_{g_i, 0}$ for all $i \in I$. Here $\mathbf{S}_{Y, 0, 0}$ is from Example 4.3 with vector bundle E and section s both zero, and $\mathbf{S}_{g_i, 0} : \mathbf{S}_{V_i, E_i, s_i} \rightarrow \mathbf{S}_{Y, 0, 0} = \mathbf{Y}$ is from Example 4.4 with $\hat{g}_i = 0$.

The hypotheses of Theorem 4.16 are similar to the notion of *good coordinate system* in the theory of Kuranishi spaces of Fukaya and Ono [11, Def. 6.1]. The importance of Theorem 4.16 is that all the ingredients are described wholly in differential-geometric or topological terms. So we can use the theorem as a tool to prove the existence of d -manifold structures on spaces coming from other areas of geometry, for instance, on moduli spaces.

4.5 Submersions, immersions and embeddings

Let $f : X \rightarrow Y$ be a smooth map of manifolds. Then $df^* : f^*(T^*Y) \rightarrow T^*X$ is a morphism of vector bundles on X , and f is a *submersion* if df^* is injective, and f is an *immersion* if df^* is surjective. Here the appropriate notions of injective and surjective for morphisms of vector bundles are stronger than the corresponding notions for sheaves: df^* is *injective* if it has a left inverse, and *surjective* if it has a right inverse.

In a similar way, if $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a 1-morphism of d-manifolds, we would like to define \mathbf{f} to be a submersion or immersion if the 1-morphism $\Omega_{\mathbf{f}} : \underline{f}^*(T^*\mathbf{Y}) \rightarrow T^*\mathbf{X}$ in $\mathbf{vvec}(\underline{X})$ is injective or surjective in some suitable sense. It turns out that there are two different notions of injective and surjective 1-morphisms in the 2-category $\mathbf{vvec}(\underline{X})$, a weak and a strong:

Definition 4.17. Let \underline{X} be a C^∞ -scheme, $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ and $(\mathcal{F}^1, \mathcal{F}^2, \psi)$ be virtual vector bundles on \underline{X} , and $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ be a 1-morphism in $\mathbf{vvec}(\underline{X})$. Then we have a complex in $\mathbf{qcoh}(\underline{X})$:

$$0 \longrightarrow \mathcal{E}^1 \xrightarrow[\gamma]{f^1 \oplus -\phi} \mathcal{F}^1 \oplus \mathcal{E}^2 \xrightarrow[\delta]{\psi \oplus f^2} \mathcal{F}^2 \longrightarrow 0. \quad (4.3)$$

One can show that f^\bullet is an equivalence in $\mathbf{vvec}(\underline{X})$ if and only if (4.3) is a *split short exact sequence* in $\mathbf{qcoh}(\underline{X})$. That is, f^\bullet is an equivalence if and only if there exist morphisms γ, δ as shown in (4.3) satisfying the conditions:

$$\begin{aligned} \gamma \circ \delta &= 0, & \gamma \circ (f^1 \oplus -\phi) &= \text{id}_{\mathcal{E}^1}, \\ (f^1 \oplus -\phi) \circ \gamma + \delta \circ (\psi \oplus f^2) &= \text{id}_{\mathcal{F}^1 \oplus \mathcal{E}^2}, & (\psi \oplus f^2) \circ \delta &= \text{id}_{\mathcal{F}^2}. \end{aligned} \quad (4.4)$$

Our notions of f^\bullet injective or surjective impose some but not all of (4.4):

- (a) We call f^\bullet *weakly injective* if there exists $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \rightarrow \mathcal{E}^1$ in $\mathbf{qcoh}(\underline{X})$ with $\gamma \circ (f^1 \oplus -\phi) = \text{id}_{\mathcal{E}^1}$.
- (b) We call f^\bullet *injective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \rightarrow \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \rightarrow \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma \circ \delta = 0$, $\gamma \circ (f^1 \oplus -\phi) = \text{id}_{\mathcal{E}^1}$ and $(f^1 \oplus -\phi) \circ \gamma + \delta \circ (\psi \oplus f^2) = \text{id}_{\mathcal{F}^1 \oplus \mathcal{E}^2}$.
- (c) We call f^\bullet *weakly surjective* if there exists $\delta : \mathcal{F}^2 \rightarrow \mathcal{F}^1 \oplus \mathcal{E}^2$ in $\mathbf{qcoh}(\underline{X})$ with $(\psi \oplus f^2) \circ \delta = \text{id}_{\mathcal{F}^2}$.
- (d) We call f^\bullet *surjective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \rightarrow \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \rightarrow \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma \circ \delta = 0$, $\gamma \circ (f^1 \oplus -\phi) = \text{id}_{\mathcal{E}^1}$ and $(\psi \oplus f^2) \circ \delta = \text{id}_{\mathcal{F}^2}$.

Using these we define weak and strong forms of submersions, immersions, and embeddings for d-manifolds.

Definition 4.18. Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a 1-morphism of d-manifolds. Definition 4.9 defines a 1-morphism $\Omega_{\mathbf{f}} : \underline{f}^*(T^*\mathbf{Y}) \rightarrow T^*\mathbf{X}$ in $\mathbf{vvec}(\underline{X})$. Then:

- (a) We call \mathbf{f} a *w-submersion* if $\Omega_{\mathbf{f}}$ is weakly injective.
- (b) We call \mathbf{f} a *submersion* if $\Omega_{\mathbf{f}}$ is injective.

- (c) We call \mathbf{f} a *w-immersion* if $\Omega_{\mathbf{f}}$ is weakly surjective.
- (d) We call \mathbf{f} an *immersion* if $\Omega_{\mathbf{f}}$ is surjective.
- (e) We call \mathbf{f} a *w-embedding* if it is a w-immersion and $f : X \rightarrow f(X)$ is a homeomorphism, so in particular f is injective.
- (f) We call \mathbf{f} an *embedding* if it is an immersion and f is a homeomorphism with its image.

Here w-submersion is short for *weak submersion*, etc. Conditions (a)–(d) all concern the existence of morphisms γ, δ in the next equation satisfying identities.

$$0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \xleftarrow[\gamma]{f'' \oplus -\underline{f}^*(\phi_Y)} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \xleftarrow[\delta]{\phi_X \oplus f^2} \mathcal{F}_X \longrightarrow 0.$$

Parts (c)–(f) enable us to define *d-submanifolds* of d-manifolds. *Open d-submanifolds* are open d-subspaces of a d-manifold. More generally, we call $\mathbf{i} : \mathbf{X} \rightarrow \mathbf{Y}$ a *w-immersed*, or *immersed*, or *w-embedded*, or *embedded d-submanifold*, of \mathbf{Y} , if \mathbf{X}, \mathbf{Y} are d-manifolds and \mathbf{i} is a w-immersion, immersion, w-embedding, or embedding, respectively.

Here are some properties of these, taken from [21, §4.1–§4.2]:

Theorem 4.19. (i) *Any equivalence of d-manifolds is a w-submersion, submersion, w-immersion, immersion, w-embedding and embedding.*

(ii) *If $\mathbf{f}, \mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$ are 2-isomorphic 1-morphisms of d-manifolds then \mathbf{f} is a w-submersion, submersion, ..., embedding, if and only if \mathbf{g} is.*

(iii) *Compositions of w-submersions, submersions, w-immersions, immersions, w-embeddings, and embeddings are 1-morphisms of the same kind.*

(iv) *The conditions that a 1-morphism of d-manifolds $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a w-submersion, submersion, w-immersion or immersion are local in \mathbf{X} and \mathbf{Y} . That is, for each $x \in \mathbf{X}$ with $\mathbf{f}(x) = y \in \mathbf{Y}$, it suffices to check the conditions for $\mathbf{f}|_U : U \rightarrow V$ with V an open neighbourhood of y in \mathbf{Y} , and U an open neighbourhood of x in $\mathbf{f}^{-1}(V) \subseteq \mathbf{X}$.*

(v) *Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a submersion of d-manifolds. Then $\text{vdim } \mathbf{X} \geq \text{vdim } \mathbf{Y}$, and if $\text{vdim } \mathbf{X} = \text{vdim } \mathbf{Y}$ then \mathbf{f} is étale.*

(vi) *Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be an immersion of d-manifolds. Then $\text{vdim } \mathbf{X} \leq \text{vdim } \mathbf{Y}$, and if $\text{vdim } \mathbf{X} = \text{vdim } \mathbf{Y}$ then \mathbf{f} is étale.*

(vii) *Let $f : X \rightarrow Y$ be a smooth map of manifolds, and $\mathbf{f} = F_{\mathbf{Man}}^{\mathbf{dMan}}(f)$. Then \mathbf{f} is a submersion, immersion, or embedding in \mathbf{dMan} if and only if f is a submersion, immersion, or embedding in \mathbf{Man} , respectively. Also \mathbf{f} is a w-immersion or w-embedding if and only if f is an immersion or embedding.*

(viii) *Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a 1-morphism of d-manifolds, with \mathbf{Y} a manifold. Then \mathbf{f} is a w-submersion.*

(ix) *Let \mathbf{X}, \mathbf{Y} be d-manifolds, with \mathbf{Y} a manifold. Then $\pi_{\mathbf{X}} : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{X}$ is a submersion.*

(x) Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a submersion of d -manifolds, and $x \in \mathbf{X}$ with $\mathbf{f}(x) = y \in \mathbf{Y}$. Then there exist open $U \subseteq \mathbf{X}$ and $V \subseteq \mathbf{Y}$ with $\mathbf{f}(U) = V$, a manifold \mathbf{Z} , and an equivalence $\mathbf{i} : U \rightarrow V \times \mathbf{Z}$, such that $\mathbf{f}|_U : U \rightarrow V$ is 2-isomorphic to $\pi_V \circ \mathbf{i}$, where $\pi_V : V \times \mathbf{Z} \rightarrow V$ is the projection.

(xi) Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a submersion of d -manifolds with \mathbf{Y} a manifold. Then \mathbf{X} is a manifold.

4.6 D-transversality and fibre products

From §3.3, if $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$ and $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ are 1-morphisms of d -manifolds then a fibre product $\mathbf{W} = \mathbf{X}_{\mathbf{g}, \mathbf{Z}, \mathbf{h}} \mathbf{Y}$ exists in \mathbf{dSpa} , and is unique up to equivalence. We want to know whether \mathbf{W} is a d -manifold. We will define when \mathbf{g}, \mathbf{h} are d -transverse, which is a sufficient condition for \mathbf{W} to be a d -manifold.

Recall that if $g : X \rightarrow Z$, $h : Y \rightarrow Z$ are smooth maps of manifolds, then a fibre product $W = X \times_{g, Z, h} Y$ in \mathbf{Man} exists if g, h are *transverse*, that is, if $T_z Z = dg|_x(T_x X) + dh|_y(T_y Y)$ for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$. Equivalently, $dg|_x^* \oplus dh|_y^* : T_x^* X \oplus T_y^* Y \rightarrow T_z^* Z$ should be injective. Writing $W = X \times_Z Y$ for the topological fibre product and $e : W \rightarrow X$, $f : W \rightarrow Y$ for the projections, with $g \circ e = h \circ f$, we see that g, h are transverse if and only if

$$e^*(dg^*) \oplus f^*(dh^*) : (g \circ e)^*(T^*Z) \rightarrow e^*(T^*X) \oplus f^*(T^*Y) \quad (4.5)$$

is an injective morphism of vector bundles on the topological space W , that is, it has a left inverse. The condition that (4.6) has a left inverse is an analogue of this, but on (dual) obstruction rather than cotangent bundles.

Definition 4.20. Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be d -manifolds and $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ be 1-morphisms. Let $\underline{W} = \underline{X} \times_{\underline{g}, \mathbf{Z}, \underline{h}} \underline{Y}$ be the C^∞ -scheme fibre product, and write $\underline{e} : \underline{W} \rightarrow \underline{X}$, $\underline{f} : \underline{W} \rightarrow \underline{Y}$ for the projections. Consider the morphism

$$\alpha = \underline{e}^*(g'') \oplus -\underline{f}^*(h'') \oplus (\underline{g} \circ \underline{e})^*(\phi_Z) : (\underline{g} \circ \underline{e})^*(\mathcal{E}_Z) \longrightarrow \underline{e}^*(\mathcal{E}_X) \oplus \underline{f}^*(\mathcal{E}_Y) \oplus (\underline{g} \circ \underline{e})^*(\mathcal{F}_Z) \quad (4.6)$$

in $\mathbf{qcoh}(\underline{W})$. We call \mathbf{g}, \mathbf{h} d -transverse if α has a left inverse.

In the notation of §4.3 and §4.5, we have 1-morphisms $\Omega_{\mathbf{g}} : \underline{g}^*(T^*\mathbf{Z}) \rightarrow T^*\mathbf{X}$ in $\mathbf{vvect}(\underline{X})$ and $\Omega_{\mathbf{h}} : \underline{h}^*(T^*\mathbf{Z}) \rightarrow T^*\mathbf{Y}$ in $\mathbf{vvect}(\underline{Y})$. Pulling these back to $\mathbf{vvect}(\underline{W})$ using $\underline{e}^*, \underline{f}^*$ we form the 1-morphism in $\mathbf{vvect}(\underline{W})$:

$$\underline{e}^*(\Omega_{\mathbf{g}}) \oplus \underline{f}^*(\Omega_{\mathbf{h}}) : (\underline{g} \circ \underline{e})^*(T^*\mathbf{Z}) \longrightarrow \underline{e}^*(T^*\mathbf{X}) \oplus \underline{f}^*(T^*\mathbf{Y}). \quad (4.7)$$

For (4.6) to have a left inverse is equivalent to (4.7) being weakly injective, as in Definition 4.17. This is the d -manifold analogue of (4.5) being injective.

Here are the main results of [21, §4.3]:

Theorem 4.21. Suppose $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are d -manifolds and $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ are d -transverse 1-morphisms, and let $\mathbf{W} = \mathbf{X}_{\mathbf{g}, \mathbf{Z}, \mathbf{h}} \mathbf{Y}$ be the d -space fibre product. Then \mathbf{W} is a d -manifold, with

$$\mathbf{vdim} \mathbf{W} = \mathbf{vdim} \mathbf{X} + \mathbf{vdim} \mathbf{Y} - \mathbf{vdim} \mathbf{Z}. \quad (4.8)$$

Theorem 4.22. Suppose $g : X \rightarrow Z$, $h : Y \rightarrow Z$ are 1-morphisms of d -manifolds. The following are sufficient conditions for g, h to be d -transverse, so that $W = X \times_{g,Z,h} Y$ is a d -manifold of virtual dimension (4.8):

- (a) Z is a manifold, that is, $Z \in \hat{\mathbf{Man}}$; or
- (b) g or h is a w -submersion.

The point here is that roughly speaking, g, h are d -transverse if they map the direct sum of the obstruction spaces of X, Y surjectively onto the obstruction spaces of Z . If Z is a manifold its obstruction spaces are zero. If g is a w -submersion it maps the obstruction spaces of X surjectively onto the obstruction spaces of Z . In both cases, d -transversality follows. See [31, Th. 8.15] for the analogue of Theorem 4.22(a) for Spivak's derived manifolds.

Theorem 4.23. Let X, Z be d -manifolds, Y a manifold, and $g : X \rightarrow Z$, $h : Y \rightarrow Z$ be 1-morphisms with g a submersion. Then $W = X \times_{g,Z,h} Y$ is a manifold, with $\dim W = \dim X + \dim Y - \dim Z$.

Theorem 4.23 shows that we may think of submersions as *representable* 1-morphisms in \mathbf{dMan} . We can locally characterize embeddings and immersions in \mathbf{dMan} in terms of fibre products with \mathbb{R}^n in \mathbf{dMan} .

Theorem 4.24. (a) Let X be a d -manifold and $g : X \rightarrow \mathbb{R}^n$ a 1-morphism in \mathbf{dMan} . Then the fibre product $W = X \times_{g,\mathbb{R}^n,0} *$ exists in \mathbf{dMan} by Theorem 4.22(a), and the projection $e : W \rightarrow X$ is an embedding.

(b) Suppose $f : X \rightarrow Y$ is an immersion of d -manifolds, and $x \in X$ with $f(x) = y \in Y$. Then there exist open d -submanifolds $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, and a 1-morphism $g : V \rightarrow \mathbb{R}^n$ with $g(y) = 0$, where $n = \dim Y - \dim X \geq 0$, fitting into a 2-Cartesian square in \mathbf{dMan} :

$$\begin{array}{ccc} U & \xrightarrow{\quad} & * \\ \downarrow f|_U & \nearrow g & \downarrow 0 \\ V & \xrightarrow{\quad} & \mathbb{R}^n. \end{array}$$

If f is an embedding we may take $U = f^{-1}(V)$.

Remark 4.25. For the applications the author has in mind, it will be crucial that if $g : X \rightarrow Z$ and $h : Y \rightarrow Z$ are 1-morphisms with X, Y d -manifolds and Z a manifold then $W = X \times_Z Y$ is a d -manifold, with $\dim W = \dim X + \dim Y - \dim Z$, as in Theorem 4.22(a). We will show by example, following Spivak [31, Prop. 1.7], that if d -manifolds \mathbf{dMan} were an ordinary category containing manifolds as a full subcategory, then this would be false.

Consider the fibre product $* \times_{0,\mathbb{R},0} *$ in \mathbf{dMan} . If \mathbf{dMan} were a category then as $*$ is a terminal object, the fibre product would be $*$. But then

$$\dim(* \times_{0,\mathbb{R},0} *) = \dim * = 0 \neq -1 = \dim * + \dim * - \dim \mathbb{R},$$

so equation (4.8) and Theorem 4.22(a) would be false. Thus, if we want fibre products of d-manifolds over manifolds to be well behaved, then \mathbf{dMan} must be at least a 2-category. It could be an ∞ -category, as for Spivak's derived manifolds [31], or some other kind of higher category. Making d-manifolds into a 2-category, as we have done, is the simplest of the available options.

4.7 Embedding d-manifolds into manifolds

Let V be a manifold, $E \rightarrow V$ a vector bundle, and $s \in C^\infty(E)$. Then Example 4.3 defines a 'standard model' principal d-manifold $\mathbf{S}_{V,E,s}$. When E and s are zero, we have $\mathbf{S}_{V,0,0} = \mathbf{V} = F_{\mathbf{Man}}^{\mathbf{dMan}}(V)$, so that $\mathbf{S}_{V,0,0}$ is a manifold. For general V, E, s , taking $f = \text{id}_V : V \rightarrow V$ and $\hat{f} = 0 : 0 \rightarrow E$ in Example 4.4 gives a 'standard model' 1-morphism $\mathbf{S}_{\text{id}_V,0} : \mathbf{S}_{V,E,s} \rightarrow \mathbf{S}_{V,0,0} = \mathbf{V}$. One can show $\mathbf{S}_{\text{id}_V,0}$ is an embedding, in the sense of Definition 4.18. Any principal d-manifold \mathbf{U} is equivalent to some $\mathbf{S}_{V,E,s}$. Thus we deduce:

Lemma 4.26. *Any principal d-manifold \mathbf{U} admits an embedding $i : \mathbf{U} \rightarrow \mathbf{V}$ into a manifold \mathbf{V} .*

Theorem 4.31 below is a converse to this: if a d-manifold \mathbf{X} can be embedded into a manifold \mathbf{Y} , then \mathbf{X} is principal. So it will be useful to study embeddings of d-manifolds into manifolds. The following facts are due to Whitney [35].

Theorem 4.27. (a) *Let X be an m -manifold and $n \geq 2m$. Then a generic smooth map $f : X \rightarrow \mathbb{R}^n$ is an immersion.*

(b) *Let X be an m -manifold and $n \geq 2m + 1$. Then there exists an embedding $f : X \rightarrow \mathbb{R}^n$, and we can choose such f with $f(X)$ closed in \mathbb{R}^n . Generic smooth maps $f : X \rightarrow \mathbb{R}^n$ are embeddings.*

In [21, §4.4] we generalize Theorem 4.27 to d-manifolds.

Theorem 4.28. *Let \mathbf{X} be a d-manifold. Then there exist immersions and/or embeddings $\mathbf{f} : \mathbf{X} \rightarrow \mathbb{R}^n$ for some $n \geq 0$ if and only if there is an upper bound for $\dim T_x^* \underline{X}$ for all $x \in \underline{X}$. If there is such an upper bound, then immersions $\mathbf{f} : \mathbf{X} \rightarrow \mathbb{R}^n$ exist provided $n \geq 2 \dim T_x^* \underline{X}$ for all $x \in \underline{X}$, and embeddings $\mathbf{f} : \mathbf{X} \rightarrow \mathbb{R}^n$ exist provided $n \geq 2 \dim T_x^* \underline{X} + 1$ for all $x \in \underline{X}$. For embeddings we may also choose \mathbf{f} with $f(X)$ closed in \mathbb{R}^n .*

Here is an example in which the condition does not hold.

Example 4.29. $\mathbb{R}^k \times_{0, \mathbb{R}^k, 0} *$ is a principal d-manifold of virtual dimension 0, with C^∞ -scheme \mathbb{R}^k , and obstruction bundle \mathbb{R}^k . Thus $\mathbf{X} = \coprod_{k \geq 0} \mathbb{R}^k \times_{0, \mathbb{R}^k, 0} *$ is a d-manifold of virtual dimension 0, with C^∞ -scheme $\underline{X} = \coprod_{k \geq 0} \mathbb{R}^k$. Since $T_x^* \underline{X} \cong \mathbb{R}^n$ for $x \in \mathbb{R}^n \subset \coprod_{k \geq 0} \mathbb{R}^k$, $\dim T_x^* \underline{X}$ realizes all values $n \geq 0$. Hence there cannot exist immersions or embeddings $\mathbf{f} : \mathbf{X} \rightarrow \mathbb{R}^n$ for any $n \geq 0$.

As $x \mapsto \dim T_x^* \underline{X}$ is an upper semicontinuous map $X \rightarrow \mathbb{N}$, if \mathbf{X} is compact then $\dim T_x^* \underline{X}$ is bounded above, giving:

Corollary 4.30. *Let X be a compact d -manifold. Then there exists an embedding $f : X \rightarrow \mathbb{R}^n$ for some $n \gg 0$.*

If a d -manifold X can be embedded into a manifold Y , we show in [21, §4.4] that we can write X as the zeroes of a section of a vector bundle over Y near its image. See [31, Prop. 9.5] for the analogue for Spivak’s derived manifolds.

Theorem 4.31. *Suppose X is a d -manifold, Y a manifold, and $f : X \rightarrow Y$ an embedding, in the sense of Definition 4.18. Then there exist an open subset V in Y with $f(X) \subseteq V$, a vector bundle $E \rightarrow V$, and $s \in C^\infty(E)$ fitting into a 2-Cartesian diagram in \mathbf{dSpa} :*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & V \\ \downarrow f & \begin{array}{c} f \\ \nearrow \eta \end{array} & \downarrow \mathbf{o} \\ V & \xrightarrow{\quad s \quad} & E. \end{array}$$

Here $Y = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, and similarly for V, E, s, \mathbf{o} , with $0 : V \rightarrow E$ the zero section. Hence X is equivalent to the ‘standard model’ d -manifold $S_{V,E,s}$ of Example 4.3, and is a principal d -manifold.

Combining Theorems 4.28 and 4.31, Lemma 4.26, and Corollary 4.30 yields:

Corollary 4.32. *Let X be a d -manifold. Then X is a principal d -manifold if and only if $\dim T_x^*X$ is bounded above for all $x \in X$. In particular, if X is compact, then X is principal.*

Corollary 4.32 suggests that most interesting d -manifolds are principal, in a similar way to most interesting C^∞ -schemes being affine in Remark 2.9(ii). Example 4.29 gives a d -manifold which is not principal.

4.8 Orientations on d -manifolds

Let X be an n -manifold. Then T^*X is a rank n vector bundle on X , so its top exterior power $\Lambda^n T^*X$ is a line bundle (rank 1 vector bundle) on X . In algebraic geometry, $\Lambda^n T^*X$ would be called the canonical bundle of X . We define an *orientation* ω on X to be an *orientation on the fibres of $\Lambda^n T^*X$* . That is, ω is an equivalence class $[\tau]$ of isomorphisms of line bundles $\tau : O_X \rightarrow \Lambda^n T^*X$, where O_X is the trivial line bundle $\mathbb{R} \times X \rightarrow X$, and τ, τ' are equivalent if $\tau' = \tau \cdot c$ for some smooth $c : X \rightarrow (0, \infty)$.

To generalize all this to d -manifolds, we will need a notion of the ‘top exterior power’ $\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}$ of a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ in §4.3. As the definition is long and complicated, we will not give it, but just state its important properties.

Theorem 4.33. *Let X be a C^∞ -scheme, and $(\mathcal{E}^\bullet, \phi)$ a virtual vector bundle on X . Then in [21, §4.5] we define a line bundle (rank 1 vector bundle) $\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}$ on X , which we call the **orientation line bundle** of $(\mathcal{E}^\bullet, \phi)$. This satisfies:*

- (a) *Suppose $\mathcal{E}^1, \mathcal{E}^2$ are vector bundles on X with ranks k_1, k_2 , and $\phi : \mathcal{E}^1 \rightarrow \mathcal{E}^2$ is a morphism. Then $(\mathcal{E}^\bullet, \phi)$ is a virtual vector bundle of rank $k_2 - k_1$, and there is a canonical isomorphism $\mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \cong \Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2} \mathcal{E}^2$.*

- (b) Let $f^\bullet : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ be an equivalence in $\text{vvect}(\underline{X})$. Then there is a canonical isomorphism $\mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \rightarrow \mathcal{L}_{(\mathcal{F}^\bullet, \psi)}$ in $\text{qcoh}(\underline{X})$.
- (c) If $(\mathcal{E}^\bullet, \phi) \in \text{vvect}(\underline{X})$ then $\mathcal{L}_{\text{id}_\phi} = \text{id}_{\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \rightarrow \mathcal{L}_{(\mathcal{E}^\bullet, \phi)}$.
- (d) If $f^\bullet : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ and $g^\bullet : (\mathcal{F}^\bullet, \psi) \rightarrow (\mathcal{G}^\bullet, \xi)$ are equivalences in $\text{vvect}(\underline{X})$ then $\mathcal{L}_{g^\bullet \circ f^\bullet} = \mathcal{L}_{g^\bullet} \circ \mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \rightarrow \mathcal{L}_{(\mathcal{G}^\bullet, \xi)}$.
- (e) If $f^\bullet, g^\bullet : (\mathcal{E}^\bullet, \phi) \rightarrow (\mathcal{F}^\bullet, \psi)$ are 2-isomorphic equivalences in $\text{vvect}(\underline{X})$ then $\mathcal{L}_{f^\bullet} = \mathcal{L}_{g^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \rightarrow \mathcal{L}_{(\mathcal{F}^\bullet, \psi)}$.
- (f) Let $\underline{f} : \underline{X} \rightarrow \underline{Y}$ be a morphism of C^∞ -schemes, and $(\mathcal{E}^\bullet, \phi) \in \text{vvect}(\underline{Y})$. Then there is a canonical isomorphism $I_{\underline{f}, (\mathcal{E}^\bullet, \phi)} : \underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}) \rightarrow \mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet, \phi)}$.

Now we can define orientations on d-manifolds.

Definition 4.34. Let \mathbf{X} be a d-manifold. Then the virtual cotangent bundle $T^*\mathbf{X}$ is a virtual vector bundle on \underline{X} by Proposition 4.10(b), so Theorem 4.33 gives a line bundle $\mathcal{L}_{T^*\mathbf{X}}$ on \underline{X} . We call $\mathcal{L}_{T^*\mathbf{X}}$ the *orientation line bundle* of \mathbf{X} .

An *orientation* ω on \mathbf{X} is an orientation on $\mathcal{L}_{T^*\mathbf{X}}$. That is, ω is an equivalence class $[\tau]$ of isomorphisms $\tau : \mathcal{O}_X \rightarrow \mathcal{L}_{T^*\mathbf{X}}$ in $\text{qcoh}(\underline{X})$, where τ, τ' are equivalent if they are proportional by a smooth positive function on \underline{X} .

If $\omega = [\tau]$ is an orientation on \mathbf{X} , the *opposite orientation* is $-\omega = [-\tau]$, which changes the sign of the isomorphism $\tau : \mathcal{O}_X \rightarrow \mathcal{L}_{T^*\mathbf{X}}$. When we refer to \mathbf{X} as an oriented d-manifold, $-\mathbf{X}$ will mean \mathbf{X} with the opposite orientation, that is, \mathbf{X} is short for (\mathbf{X}, ω) and $-\mathbf{X}$ is short for $(\mathbf{X}, -\omega)$.

Example 4.35. (a) Let X be an n -manifold, and $\mathbf{X} = F_{\text{Man}}^{\text{dMan}}(X)$ the associated d-manifold. Then $\underline{X} = F_{\text{Man}}^{C^\infty \text{Sch}}(X)$, $\mathcal{E}_X = 0$ and $\mathcal{F}_X = T^*\underline{X}$. So $\mathcal{E}_X, \mathcal{F}_X$ are vector bundles of ranks $0, n$. As $\Lambda^0 \mathcal{E}_X \cong \mathcal{O}_X$, Theorem 4.33(a) gives a canonical isomorphism $\mathcal{L}_{T^*\mathbf{X}} \cong \Lambda^n T^*\underline{X}$. That is, $\mathcal{L}_{T^*\mathbf{X}}$ is isomorphic to the lift to C^∞ -schemes of the line bundle $\Lambda^n T^*X$ on the manifold X .

As above, an orientation on X is an orientation on the line bundle $\Lambda^n T^*X$. Hence orientations on the d-manifold $\mathbf{X} = F_{\text{Man}}^{\text{dMan}}(X)$ in the sense of Definition 4.34 are equivalent to orientations on the manifold X in the usual sense.

(b) Let V be an n -manifold, $E \rightarrow V$ a vector bundle of rank k , and $s \in C^\infty(E)$. Then Example 4.3 defines a ‘standard model’ principal d-manifold $\mathbf{S} = \mathbf{S}_{V, E, s}$, which has $\mathcal{E}_\mathbf{S} \cong \underline{E}^*|_\mathbf{S}$, $\mathcal{F}_\mathbf{S} \cong T^*\underline{V}|_\mathbf{S}$, where $\underline{E}, T^*\underline{V}$ are the lifts of the vector bundles E, T^*V on V to \underline{V} . Hence $\mathcal{E}_\mathbf{S}, \mathcal{F}_\mathbf{S}$ are vector bundles on $\underline{S}_{V, E, s}$ of ranks k, n , so Theorem 4.33(a) gives an isomorphism $\mathcal{L}_{T^*\mathbf{S}_{V, E, s}} \cong (\Lambda^k \underline{E} \otimes \Lambda^n T^*\underline{V})|_\mathbf{S}$.

Thus $\mathcal{L}_{T^*\mathbf{S}_{V, E, s}}$ is the lift to $\underline{S}_{V, E, s}$ of the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ over the manifold V . Therefore we may induce an orientation on the d-manifold $\mathbf{S}_{V, E, s}$ from an orientation on the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ over V . Equivalently, we can induce an orientation on $\mathbf{S}_{V, E, s}$ from an orientation on the total space of the vector bundle E^* over V , or from an orientation on the total space of E .

We can construct orientations on d-transverse fibre products of oriented d-manifolds. Note that (4.9) depends on an *orientation convention*: a different choice would change (4.9) by a sign depending on $\text{vdim } \mathbf{X}, \text{vdim } \mathbf{Y}, \text{vdim } \mathbf{Z}$. Our conventions follow those of Fukaya et al. [10, §8.2] for Kuranishi spaces.

Theorem 4.36. *Work in the situation of Theorem 4.21, so that W, X, Y, Z are d -manifolds with $W = X \times_{g, Z, h} Y$ for g, h d -transverse, where $e : W \rightarrow X$, $f : W \rightarrow Y$ are the projections. Then we have orientation line bundles $\mathcal{L}_{T^*W}, \dots, \mathcal{L}_{T^*Z}$ on $\underline{W}, \dots, \underline{Z}$, so $\mathcal{L}_{T^*W}, \underline{e}^*(\mathcal{L}_{T^*X}), \underline{f}^*(\mathcal{L}_{T^*Y}), (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*Z})$ are line bundles on \underline{W} . With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Phi : \mathcal{L}_{T^*W} \longrightarrow \underline{e}^*(\mathcal{L}_{T^*X}) \otimes_{\mathcal{O}_W} \underline{f}^*(\mathcal{L}_{T^*Y}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(\mathcal{L}_{T^*Z})^*. \quad (4.9)$$

Hence, if X, Y, Z are oriented d -manifolds, then W also has a natural orientation, since trivializations of $\mathcal{L}_{T^*X}, \mathcal{L}_{T^*Y}, \mathcal{L}_{T^*Z}$ induce a trivialization of \mathcal{L}_{T^*W} by (4.9).

Fibre products have natural commutativity and associativity properties. When we include orientations, the orientations differ by some sign. Here is an analogue of results of Fukaya et al. [10, Lem. 8.2.3] for Kuranishi spaces.

Proposition 4.37. *Suppose V, \dots, Z are oriented d -manifolds, e, \dots, h are 1-morphisms, and all fibre products below are d -transverse. Then the following hold, in oriented d -manifolds:*

(a) *For $g : X \rightarrow Z$ and $h : Y \rightarrow Z$ we have*

$$X \times_{g, Z, h} Y \simeq (-1)^{(\text{vdim } X - \text{vdim } Z)(\text{vdim } Y - \text{vdim } Z)} Y \times_{h, Z, g} X.$$

*In particular, when $Z = *$ so that $X \times_Z Y = X \times Y$ we have*

$$X \times Y \simeq (-1)^{\text{vdim } X \text{vdim } Y} Y \times X.$$

(b) *For $e : V \rightarrow Y$, $f : W \rightarrow Y$, $g : W \rightarrow Z$, and $h : X \rightarrow Z$ we have*

$$V \times_{e, Y, f \circ \pi_W} (W \times_{g, Z, h} X) \simeq (V \times_{e, Y, f} W) \times_{g \circ \pi_W, Z, h} X.$$

(c) *For $e : V \rightarrow Y$, $f : V \rightarrow Z$, $g : W \rightarrow Y$, and $h : X \rightarrow Z$ we have*

$$\begin{aligned} V \times_{(e, f), Y \times Z, g \times h} (W \times X) &\simeq \\ (-1)^{\text{vdim } Z(\text{vdim } Y + \text{vdim } W)} (V \times_{e, Y, g} W) &\times_{f \circ \pi_V, Z, h} X. \end{aligned}$$

4.9 D-manifolds with boundary and corners, d-orbifolds

For brevity, this section will give much less detail than §4.1–§4.8. So far we have discussed only manifolds *without boundary* (locally modelled on \mathbb{R}^n). One can also consider *manifolds with boundary* (locally modelled on $[0, \infty) \times \mathbb{R}^{n-1}$) and *manifolds with corners* (locally modelled on $[0, \infty)^k \times \mathbb{R}^{n-k}$). In [18] the author studied manifolds with boundary and with corners, giving a new definition of *smooth map* $f : X \rightarrow Y$ between manifolds with corners X, Y , satisfying extra

conditions over $\partial^k X, \partial^l Y$. This yields categories $\mathbf{Man}^b, \mathbf{Man}^c$ of manifolds with boundary and with corners with good properties *as categories*.

In [21, §6–§7], the author defined 2-categories $\mathbf{dSpa}^b, \mathbf{dSpa}^c$ of *d-spaces with boundary* and *with corners*, and 2-subcategories $\mathbf{dMan}^b, \mathbf{dMan}^c$ of *d-manifolds with boundary* and *with corners*. Objects in $\mathbf{dSpa}^b, \mathbf{dSpa}^c, \mathbf{dMan}^b, \mathbf{dMan}^c$ are quadruples $\mathbf{X} = (X, \partial X, i_X, \omega_X)$, where $X, \partial X$ are d-spaces, and $i_X : \partial X \rightarrow X$ is a 1-morphism, such that ∂X is locally equivalent to a fibre product $X \times_{[0, \infty)} *$ in \mathbf{dSpa} , in a similar way to Theorem 4.24(b). This implies that the ‘conormal bundle’ \mathcal{N}_X of ∂X in X is a line bundle on $\underline{\partial X}$. The final piece of data ω_X is an orientation on \mathcal{N}_X , giving a notion of ‘outward-pointing normal vectors’ to ∂X in X . Here are some properties of these:

Theorem 4.38. *In [21, §7] we define strict 2-categories $\mathbf{dMan}^b, \mathbf{dMan}^c$ of d-manifolds with boundary and d-manifolds with corners. These have the following properties:*

- (a) \mathbf{dMan}^b is a full 2-subcategory of \mathbf{dMan}^c . There is a full and faithful 2-functor $F_{\mathbf{dMan}}^{\mathbf{dMan}^c} : \mathbf{dMan} \hookrightarrow \mathbf{dMan}^c$ whose image is a full 2-subcategory \mathbf{dMan} in \mathbf{dMan}^b , so that $\mathbf{dMan} \subset \mathbf{dMan}^b \subset \mathbf{dMan}^c$.
- (b) There are full and faithful 2-functors $F_{\mathbf{Man}^b}^{\mathbf{dMan}^b} : \mathbf{Man}^b \rightarrow \mathbf{dMan}^b$ and $F_{\mathbf{Man}^c}^{\mathbf{dMan}^c} : \mathbf{Man}^c \rightarrow \mathbf{dMan}^c$. We write $\bar{\mathbf{Man}}^b, \bar{\mathbf{Man}}^c$ for the full 2-subcategories of objects in $\mathbf{dMan}^b, \mathbf{dMan}^c$ equivalent to objects in the images of $F_{\mathbf{Man}^b}^{\mathbf{dMan}^b}, F_{\mathbf{Man}^c}^{\mathbf{dMan}^c}$.
- (c) Each object $\mathbf{X} = (X, \partial X, i_X, \omega_X)$ in \mathbf{dMan}^b or \mathbf{dMan}^c has a **virtual dimension** $\text{vdim } \mathbf{X} \in \mathbb{Z}$. The virtual cotangent sheaf T^*X of the underlying d-space X is a virtual vector bundle on \underline{X} with rank $\text{vdim } \mathbf{X}$.
- (d) Each d-manifold with corners \mathbf{X} has a **boundary** ∂X , which is another d-manifold with corners with $\text{vdim } \partial X = \text{vdim } \mathbf{X} - 1$. The d-space 1-morphism i_X in \mathbf{X} is also a 1-morphism $i_X : \partial X \rightarrow X$ in \mathbf{dMan}^c . If $\mathbf{X} \in \mathbf{dMan}^b$ then $\partial X \in \mathbf{dMan}$, and if $\mathbf{X} \in \mathbf{dMan}$ then $\partial X = \emptyset$.
- (e) Boundaries in $\mathbf{dMan}^b, \mathbf{dMan}^c$ have strong functorial properties. For instance, there is an interesting class of **simple** 1-morphisms $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ in \mathbf{dMan}^b and \mathbf{dMan}^c , which satisfy a discrete condition broadly saying that \mathbf{f} maps $\partial^k X$ to $\partial^k Y$ for all k . These have the property that for all simple $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ there is a unique simple 1-morphism $\mathbf{f}_- : \partial X \rightarrow \partial Y$ with $\mathbf{f} \circ i_X = i_Y \circ \mathbf{f}_-$, and the following diagram is 2-Cartesian in \mathbf{dMan}^c

$$\begin{array}{ccc} \partial X & \xrightarrow{\quad \mathbf{f}_- \quad} & \partial Y \\ i_X \downarrow & \begin{array}{c} \mathbf{f}_- \quad \text{id}_{\mathbf{f} \circ i_X} \uparrow \\ \mathbf{f} \end{array} & \downarrow i_Y \\ X & \xrightarrow{\quad \mathbf{f} \quad} & Y, \end{array}$$

so that $\partial X \simeq X \times_{\mathbf{f}, Y, i_Y} \partial Y$ in \mathbf{dMan}^c . If $\mathbf{f}, \mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$ are simple 1-morphisms and $\eta : \mathbf{f} \Rightarrow \mathbf{g}$ is a 2-morphism in \mathbf{dMan}^c then there is a natural 2-morphism $\eta_- : \mathbf{f}_- \Rightarrow \mathbf{g}_-$ in \mathbf{dMan}^c .

- (f) An **orientation** on a d -manifold with corners $\mathbf{X} = (\mathbf{X}, \partial\mathbf{X}, \mathbf{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ is an orientation on the line bundle $\mathcal{L}_{T^*\mathbf{X}}$ on $\underline{\mathbf{X}}$. If \mathbf{X} is an oriented d -manifold with corners, there is a natural orientation on $\partial\mathbf{X}$, constructed using the orientation on \mathbf{X} and the data $\omega_{\mathbf{X}}$ in \mathbf{X} .
- (g) Almost all the results of §4.1–§4.8 on d -manifolds without boundary extend to d -manifolds with boundary and with corners, with some changes.

One moral of [18] and [21, §5–§7] is that doing ‘things with corners’ properly is a great deal more complicated, but also more interesting, than you would believe if you had not thought about the issues involved.

Example 4.39. (i) Let \mathbf{X} be the fibre product $[0, \infty) \times_{i, \mathbb{R}, 0} *$ in \mathbf{dMan}^c , where $i : [0, \infty) \hookrightarrow \mathbb{R}$ is the inclusion. Then $\mathbf{X} = (\mathbf{X}, \partial\mathbf{X}, \mathbf{i}_{\mathbf{X}}, \omega_{\mathbf{X}})$ is ‘a point with point boundary’, of virtual dimension 0, and its boundary $\partial\mathbf{X}$ is an ‘obstructed point’, a point with obstruction space \mathbb{R} , of virtual dimension -1 .

The conormal bundle $\mathcal{N}_{\mathbf{X}}$ of $\partial\mathbf{X}$ in \mathbf{X} is the obstruction space \mathbb{R} of $\partial\mathbf{X}$. In this case, the orientation $\omega_{\mathbf{X}}$ on $\mathcal{N}_{\mathbf{X}}$ cannot be determined from $\mathbf{X}, \partial\mathbf{X}, \mathbf{i}_{\mathbf{X}}$, in fact, there is an automorphism of $\mathbf{X}, \partial\mathbf{X}, \mathbf{i}_{\mathbf{X}}$ which reverses the orientation of $\mathcal{N}_{\mathbf{X}}$. So $\omega_{\mathbf{X}}$ really is extra data. We include $\omega_{\mathbf{X}}$ in the definition of d -manifolds with corners to ensure that orientations of d -manifolds with corners are well-behaved. If we omitted $\omega_{\mathbf{X}}$ from the definition, there would exist oriented d -manifolds with corners \mathbf{X} whose boundaries $\partial\mathbf{X}$ are not orientable.

(ii) The fibre product $[0, \infty) \times_{i, [0, \infty), 0} *$ is a point $*$ without boundary. The only difference with (i) is that we have replaced the target \mathbb{R} with $[0, \infty)$, adding a boundary. So in a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ in \mathbf{dMan}^c , the boundary of \mathbf{Z} affects the boundary of \mathbf{W} . This does not happen for fibre products in \mathbf{Man}^c .

(iii) Let \mathbf{X}' be the fibre product $[0, \infty) \times_{i, \mathbb{R}, i} (-\infty, 0]$ in \mathbf{dMan}^c , that is, the derived intersection of submanifolds $[0, \infty), (-\infty, 0]$ in \mathbb{R} . Topologically, \mathbf{X}' is just the point $\{0\}$, but as a d -manifold with corners \mathbf{X}' has virtual dimension 1. The boundary $\partial\mathbf{X}'$ is the disjoint union of two copies of \mathbf{X} in (i). The C^∞ -scheme $\underline{\mathbf{X}}$ in \mathbf{X} is the spectrum of the C^∞ -ring $C^\infty([0, \infty)^2)/(x + y)$, which is infinite-dimensional, although its topological space is a point.

Orbifolds are generalizations of manifolds locally modelled on \mathbb{R}^n/G for G a finite group. They are related to manifolds as Deligne–Mumford stacks are related to schemes in algebraic geometry, and form a 2-category **Orb**. Lerman [24] surveys definitions of orbifolds, and explains why **Orb** is a 2-category. As for **Man^b**, **Man^c** one can also consider 2-categories of *orbifolds with boundary* **Orb^b** and *orbifolds with corners* **Orb^c**, discussed in [21, §8].

In [21, §9 & §11] we define 2-categories of *d-stacks* **dSta**, *d-stacks with boundary* **dSta^b** and *d-stacks with corners* **dSta^c**, which are orbifold versions of **dSpa**, **dSpa^b**, **dSpa^c**. Broadly, to go from d -spaces $\mathbf{X} = (\underline{\mathbf{X}}, \mathcal{O}'_{\mathbf{X}}, \mathcal{E}_{\mathbf{X}}, \iota_{\mathbf{X}}, \jmath_{\mathbf{X}})$ to d -stacks we just replace the C^∞ -scheme $\underline{\mathbf{X}}$ by a *Deligne–Mumford C^∞ -stack* \mathcal{X} , where the theory of the 2-category of C^∞ -stacks **C[∞]Sta** is developed in [19, §7–§11] and summarized in [20, §4]. Then in [21, §10 & §12] we define 2-categories

of *d-orbifolds* \mathbf{dOrb} , *d-orbifolds with boundary* \mathbf{dOrb}^b and *d-orbifolds with corners* \mathbf{Orb}^c , which are orbifold versions of \mathbf{dMan} , \mathbf{dMan}^b , \mathbf{dMan}^c .

One might expect that combining the 2-categories \mathbf{Orb} and \mathbf{dMan} should result in a 3-category \mathbf{dOrb} , but in fact a 2-category is sufficient. For 1-morphisms $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ in \mathbf{dOrb} , a 2-morphism $\eta : f \Rightarrow g$ in \mathbf{dOrb} is a pair (η, η') , where $\eta : f \Rightarrow g$ is a 2-morphism in $\mathbf{C}^\infty\mathbf{Sta}$, and $\eta' : f^*(\mathcal{F}_{\mathcal{Y}}) \rightarrow \mathcal{E}_{\mathcal{X}}$ is as for 2-morphisms in \mathbf{dMan} . These η, η' do not interact very much.

The generalizations to d-orbifolds are mostly straightforward, with few surprises. Almost all the results of §3–§4.8, and Theorem 4.38, extend to d-stacks and d-orbifolds with only cosmetic changes. One exception is that the generalizations of Theorems 3.7 and 4.16 to d-stacks and d-orbifolds need extra conditions on the C^∞ -stack 2-morphism components η_{ijk} of $\boldsymbol{\eta}_{ijk}$ on quadruple overlaps $\mathcal{X}_i \cap \mathcal{X}_j \cap \mathcal{X}_k \cap \mathcal{X}_l$, as in Remark 3.8. This is because 2-morphisms η_{ijk} in $\mathbf{C}^\infty\mathbf{Sta}$ are discrete, and cannot be glued using partitions of unity.

4.10 D-manifold bordism, and virtual cycles

Classical bordism groups $MSO_k(Y)$ were defined by Atiyah [1] for topological spaces Y , using continuous maps $f : X \rightarrow Y$ for X a compact oriented manifold. Conner [8, §I] gives a good introduction. We define bordism $B_k(Y)$ only for manifolds Y , using smooth $f : X \rightarrow Y$, following Conner’s *differential bordism groups* [8, §I.9]. By [8, Th. I.9.1], the natural projection $B_k(Y) \rightarrow MSO_k(Y)$ is an isomorphism, so our notion of bordism agrees with the usual definition.

Definition 4.40. Let Y be a manifold without boundary, and $k \in \mathbb{Z}$. Consider pairs (X, f) , where X is a compact, oriented manifold without boundary with $\dim X = k$, and $f : X \rightarrow Y$ is a smooth map. Define an equivalence relation \sim on such pairs by $(X, f) \sim (X', f')$ if there exists a compact, oriented $(k+1)$ -manifold with boundary W , a smooth map $e : W \rightarrow Y$, and a diffeomorphism of oriented manifolds $j : -X \amalg X' \rightarrow \partial W$, such that $f \amalg f' = e \circ i_W \circ j$, where $-X$ is X with the opposite orientation.

Write $[X, f]$ for the \sim -equivalence class (*bordism class*) of a pair (X, f) . For each $k \in \mathbb{Z}$, define the k^{th} *bordism group* $B_k(Y)$ of Y to be the set of all such bordism classes $[X, f]$ with $\dim X = k$. We give $B_k(Y)$ the structure of an abelian group, with zero element $0_Y = [\emptyset, \emptyset]$, and addition given by $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$, and additive inverses $-[X, f] = [-X, f]$.

Define $\Pi_{\text{bo}}^{\text{hom}} : B_k(Y) \rightarrow H_k(Y; \mathbb{Z})$ by $\Pi_{\text{bo}}^{\text{hom}} : [X, f] \mapsto f_*([X])$, where $H_*(-; \mathbb{Z})$ is singular homology, and $[X] \in H_k(X; \mathbb{Z})$ is the fundamental class.

If Y is oriented and of dimension n , there is a biadditive, associative, supercommutative *intersection product* $\bullet : B_k(Y) \times B_l(Y) \rightarrow B_{k+l-n}(Y)$, such that if $[X, f], [X', f']$ are classes in $B_*(Y)$, with f, f' transverse, then the fibre product $X \times_{f, Y, f'} X'$ exists as a compact oriented manifold, and

$$[X, f] \bullet [X', f'] = [X \times_{f, Y, f'} X', f \circ \pi_X].$$

As in [8, §I.5], bordism is a generalized homology theory. Results of Thom, Wall and others in [8, §I.2] compute the bordism groups $B_k(*)$ of the point $*$. This partially determines the bordism groups of general manifolds Y , as there is a spectral sequence $H_i(Y; B_j(*)) \Rightarrow B_{i+j}(Y)$. We define *d-manifold bordism* by replacing manifolds X in $[X, f]$ by d-manifolds \mathbf{X} :

Definition 4.41. Let Y be a manifold without boundary, and $k \in \mathbb{Z}$. Consider pairs (\mathbf{X}, \mathbf{f}) , where $\mathbf{X} \in \mathbf{dMan}$ is a compact, oriented d-manifold without boundary with $\text{vdim } \mathbf{X} = k$, and $\mathbf{f} : \mathbf{X} \rightarrow Y$ is a 1-morphism in \mathbf{dMan} , where $Y = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$.

Define an equivalence relation \sim between such pairs by $(\mathbf{X}, \mathbf{f}) \sim (\mathbf{X}', \mathbf{f}')$ if there exists a compact, oriented d-manifold with boundary \mathbf{W} with $\text{vdim } \mathbf{W} = k + 1$, a 1-morphism $e : \mathbf{W} \rightarrow Y$ in \mathbf{dMan}^b , an equivalence of oriented d-manifolds $j : -\mathbf{X} \amalg \mathbf{X}' \rightarrow \partial \mathbf{W}$, and a 2-morphism $\eta : \mathbf{f} \amalg \mathbf{f}' \Rightarrow e \circ i_{\mathbf{W}} \circ j$.

Write $[\mathbf{X}, \mathbf{f}]$ for the \sim -equivalence class (*d-bordism class*) of a pair (\mathbf{X}, \mathbf{f}) . For each $k \in \mathbb{Z}$, define the k^{th} *d-manifold bordism group*, or *d-bordism group*, $dB_k(Y)$ of Y to be the set of all such d-bordism classes $[\mathbf{X}, \mathbf{f}]$ with $\text{vdim } \mathbf{X} = k$. As for $B_k(Y)$, we give $dB_k(Y)$ the structure of an abelian group, with zero element $0_Y = [\emptyset, \emptyset]$, addition $[\mathbf{X}, \mathbf{f}] + [\mathbf{X}', \mathbf{f}'] = [\mathbf{X} \amalg \mathbf{X}', \mathbf{f} \amalg \mathbf{f}']$, and additive inverses $-[\mathbf{X}, \mathbf{f}] = [-\mathbf{X}, \mathbf{f}]$.

If Y is oriented and of dimension n , we define a biadditive, associative, supercommutative *intersection product* $\bullet : dB_k(Y) \times dB_l(Y) \rightarrow dB_{k+l-n}(Y)$ by

$$[\mathbf{X}, \mathbf{f}] \bullet [\mathbf{X}', \mathbf{f}'] = [\mathbf{X} \times_{f, Y, f'} \mathbf{X}', \mathbf{f} \circ \pi_{\mathbf{X}}].$$

Here $\mathbf{X} \times_{f, Y, f'} \mathbf{X}'$ exists as a d-manifold by Theorem 4.22(a), and is oriented by Theorem 4.36. Note that we do not need to restrict to $[X, f], [X', f']$ with f, f' transverse as in Definition 4.40. Define a morphism $\Pi_{\text{bo}}^{\text{dbo}} : B_k(Y) \rightarrow dB_k(Y)$ for $k \geq 0$ by $\Pi_{\text{bo}}^{\text{dbo}} : [X, f] \mapsto [F_{\mathbf{Man}}^{\mathbf{dMan}}(X), F_{\mathbf{Man}}^{\mathbf{dMan}}(f)]$.

In [21, §13.2] we prove that $B_*(Y)$ and $dB_*(Y)$ are isomorphic. See [31, Th. 2.6] for the analogous result for Spivak's derived manifolds.

Theorem 4.42. *For any manifold Y , we have $dB_k(Y) = 0$ for $k < 0$, and $\Pi_{\text{bo}}^{\text{dbo}} : B_k(Y) \rightarrow dB_k(Y)$ is an isomorphism for $k \geq 0$. When Y is oriented, $\Pi_{\text{bo}}^{\text{dbo}}$ identifies the intersection products \bullet on $B_*(Y)$ and $dB_*(Y)$.*

Here is the main idea in the proof of Theorem 4.42. Let $[\mathbf{X}, \mathbf{f}] \in dB_k(Y)$. By Corollary 4.30 there exists an embedding $\mathbf{g} : \mathbf{X} \rightarrow \mathbb{R}^n$ for $n \gg 0$. Then the direct product $(\mathbf{f}, \mathbf{g}) : \mathbf{X} \rightarrow Y \times \mathbb{R}^n$ is also an embedding. Theorem 4.31 shows that there exist an open set $V \subseteq Y \times \mathbb{R}^n$, a vector bundle $E \rightarrow V$ and $s \in C^\infty(E)$ such that $\mathbf{X} \simeq S_{V, E, s}$. Let $\tilde{s} \in C^\infty(E)$ be a small, generic perturbation of s . As \tilde{s} is generic, the graph of \tilde{s} in E intersects the zero section transversely. Hence $\tilde{X} = \tilde{s}^{-1}(0)$ is a k -manifold for $k \geq 0$, which is compact and oriented for $\tilde{s} - s$ small, and $\tilde{X} = \emptyset$ for $k < 0$. Set $\tilde{f} = \pi_Y|_{\tilde{X}} : \tilde{X} \rightarrow Y$. Then $\Pi_{\text{bo}}^{\text{dbo}}([\tilde{X}, \tilde{f}]) = [\mathbf{X}, \mathbf{f}]$, so that $\Pi_{\text{bo}}^{\text{dbo}}$ is surjective. A similar argument for \mathbf{W}, e in Definition 4.41 shows that $\Pi_{\text{bo}}^{\text{dbo}}$ is injective.

By Theorem 4.42, we may define a projection $\Pi_{\text{dbo}}^{\text{hom}} : dB_k(Y) \rightarrow H_k(Y; \mathbb{Z})$ for $k \geq 0$ by $\Pi_{\text{dbo}}^{\text{hom}} = \Pi_{\text{bo}}^{\text{hom}} \circ (\Pi_{\text{bo}}^{\text{dbo}})^{-1}$. We think of $\Pi_{\text{dbo}}^{\text{hom}}$ as a *virtual class map*. Virtual classes (or virtual cycles, or virtual chains) are used in several areas of geometry to construct enumerative invariants using moduli spaces. In algebraic geometry, Behrend and Fantechi [4] construct virtual classes for schemes with obstruction theories. In symplectic geometry, there are many versions — see for example Fukaya et al. [11, §6], [10, §A1], Hofer et al. [14], and McDuff [27].

The main message we want to draw from this is that *oriented d-manifolds and d-orbifolds admit virtual classes* (or virtual cycles, or virtual chains, as appropriate). Thus, we can use d-manifolds and d-orbifolds as the geometric structure on moduli spaces in enumerative invariants problems such as Gromov–Witten invariants, Lagrangian Floer cohomology, Donaldson–Thomas invariants, ..., as this structure is strong enough to contain all the ‘counting’ information.

In future work the author intends to define a virtual chain construction for d-manifolds and d-orbifolds, expressed in terms of new (co)homology theories whose (co)chains are built from d-manifolds or d-orbifolds, as for the ‘Kuranishi (co)homology’ described in [17].

4.11 Relation to other classes of spaces in mathematics

In [21, §14] the author studied the relationships between d-manifolds and d-orbifolds and other classes of geometric spaces in the literature. The next theorem summarizes our results:

Theorem 4.43. *We may construct ‘truncation functors’ from various classes of geometric spaces to d-manifolds and d-orbifolds, as follows:*

- (a) *There is a functor $\Pi_{\text{BManFS}}^{\text{dMan}} : \text{BManFS} \rightarrow \text{Ho}(\text{dMan})$, where **BManFS** is a category whose objects are triples (V, E, s) of a Banach manifold V , Banach vector bundle $E \rightarrow V$, and smooth section $s : V \rightarrow E$ whose linearization $ds|_x : T_x V \rightarrow E|_x$ is Fredholm with index $n \in \mathbb{Z}$ for each $x \in V$ with $s|_x = 0$, and $\text{Ho}(\text{dMan})$ is the homotopy category of the 2-category of d-manifolds **dMan**.*

There is also an orbifold version $\Pi_{\text{BOrbFS}}^{\text{dOrb}} : \text{Ho}(\text{BOrbFS}) \rightarrow \text{Ho}(\text{dOrb})$ of this using Banach orbifolds V , and ‘corners’ versions of both.

- (b) *There is a functor $\Pi_{\text{MPolFS}}^{\text{dMan}} : \text{MPolFS} \rightarrow \text{Ho}(\text{dMan})$, where **MPolFS** is a category whose objects are triples (V, E, s) of an **M-polyfold** without boundary V as in Hofer, Wysocki and Zehnder [13, §3.3], a fillable strong M-polyfold bundle E over V [13, §4.3], and an sc-smooth Fredholm section s of E [13, §4.4] whose linearization $ds|_x : T_x V \rightarrow E|_x$ [13, §4.4] has Fredholm index $n \in \mathbb{Z}$ for all $x \in V$ with $s|_x = 0$.*

*There is also an orbifold version $\Pi_{\text{PolFS}}^{\text{dOrb}} : \text{Ho}(\text{PolFS}) \rightarrow \text{Ho}(\text{dOrb})$ of this using **polyfolds** V , and ‘corners’ versions of both.*

- (c) *Given a d-orbifold with corners \mathcal{X} , we can construct a **Kuranishi space** (X, κ) in the sense of Fukaya, Oh, Ohta and Ono [10, §A], with the same underlying topological space X . Conversely, given a Kuranishi space*

(X, κ) , we can construct a d -orbifold with corners \mathcal{X}' . Composing the two constructions, \mathcal{X} and \mathcal{X}' are equivalent in \mathbf{dOrb}^c .

Very roughly speaking, this means that the ‘categories’ of d -orbifolds with corners, and Kuranishi spaces, are equivalent. However, Fukaya et al. [10] do not define morphisms of Kuranishi spaces, nor even when two Kuranishi spaces are ‘the same’, so we have no category of Kuranishi spaces.

- (d) There is a functor $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}} : \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs} \rightarrow \mathbf{Ho}(\mathbf{dMan})$, where $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ is a category whose objects are triples (X, E^\bullet, ϕ) , for X a separated, second countable \mathbb{C} -scheme and $\phi : E^\bullet \rightarrow \tau_{\geq -1}(L_X)$ a perfect obstruction theory on X with constant virtual dimension, in the sense of Behrend and Fantechi [4]. We may define a natural orientation on $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}}(X, E^\bullet, \phi)$ for each (X, E^\bullet, ϕ) .

There is also an orbifold version $\Pi_{\mathbf{StaObs}}^{\mathbf{dOrb}} : \mathbf{Ho}(\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}) \rightarrow \mathbf{Ho}(\mathbf{dOrb})$, taking X to be a Deligne–Mumford \mathbb{C} -stack.

- (e) There is a functor $\Pi_{\mathbf{QsDSch}}^{\mathbf{dMan}} : \mathbf{Ho}(\mathbf{QsDSch}_{\mathbb{C}}) \rightarrow \mathbf{Ho}(\mathbf{dMan})$, where $\mathbf{QsDSch}_{\mathbb{C}}$ is the ∞ -category of separated, second countable, quasi-smooth derived \mathbb{C} -schemes X of constant dimension, as in Toën and Vezzosi [33, 34]. We may define a natural orientation on $\Pi_{\mathbf{QsDSch}}^{\mathbf{dMan}}(X)$ for each X .

There is also an orbifold version $\Pi_{\mathbf{QsDSta}}^{\mathbf{dOrb}} : \mathbf{Ho}(\mathbf{QsDSta}_{\mathbb{C}}) \rightarrow \mathbf{Ho}(\mathbf{dOrb})$, taking X to be a derived Deligne–Mumford \mathbb{C} -stack.

- (f) (Borisov [5]) There is a natural functor $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}} : \mathbf{Ho}(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}) \rightarrow \mathbf{Ho}(\mathbf{dMan}_{\mathbf{pr}})$ from the homotopy category of the ∞ -category $\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}}$ of **derived manifolds** of finite type with pure dimension, in the sense of Spivak [31], to the homotopy category of the full 2-subcategory $\mathbf{dMan}_{\mathbf{pr}}$ of principal d -manifolds in \mathbf{dMan} . This functor induces a bijection between isomorphism classes of objects in $\mathbf{Ho}(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}})$ and $\mathbf{Ho}(\mathbf{dMan}_{\mathbf{pr}})$. It is full, but not faithful. If $[f]$ is a morphism in $\mathbf{Ho}(\mathbf{DerMan}_{\mathbf{ft}}^{\mathbf{pd}})$, then $[f]$ is an isomorphism if and only if $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}([f])$ is an isomorphism.

One moral of Theorem 4.43 is that essentially every geometric structure on moduli spaces which is used to define enumerative invariants, either in differential geometry, or in algebraic geometry over \mathbb{C} , has a truncation functor to d -manifolds or d -orbifolds. Combining Theorem 4.43 with proofs from the literature of the existence on moduli spaces of the geometric structures listed in Theorem 4.43, in [21, §14] we deduce:

Theorem 4.44. (i) *Any solution set of a smooth nonlinear elliptic equation with fixed topological invariants on a compact manifold naturally has the structure of a d -manifold, uniquely up to equivalence in \mathbf{dMan} .*

For example, let $(M, g), (N, h)$ be Riemannian manifolds, with M compact. Then the family of **harmonic maps** $f : M \rightarrow N$ is a d -manifold $\mathcal{H}_{M,N}$ with $\text{vdim } \mathcal{H}_{M,N} = 0$. If $M = \mathcal{S}^1$, then $\mathcal{H}_{M,N}$ is the moduli space of **parametrized closed geodesics** in (N, h) .

- (ii) Let (M, ω) be a compact symplectic manifold of dimension $2n$, and J an almost complex structure on M compatible with ω . For $\beta \in H_2(M, \mathbb{Z})$ and $g, m \geq 0$, write $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ for the moduli space of stable triples (Σ, \vec{z}, u) for Σ a genus g prestable Riemann surface with m marked points $\vec{z} = (z_1, \dots, z_m)$ and $u : \Sigma \rightarrow M$ a J -holomorphic map with $[u(\Sigma)] = \beta$ in $H_2(M, \mathbb{Z})$. Using results of Hofer, Wysocki and Zehnder [15] involving their theory of polyfolds, we can make $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ into a compact, oriented d -orbifold $\mathbf{\overline{\mathcal{M}}}_{g,m}(M, J, \beta)$.
- (iii) Let (M, ω) be a compact symplectic manifold, J an almost complex structure on M compatible with ω , and L a compact, embedded Lagrangian submanifold in M . For $\beta \in H_2(M, L; \mathbb{Z})$ and $k \geq 0$, write $\overline{\mathcal{M}}_k(M, L, J, \beta)$ for the moduli space of **J -holomorphic stable maps** (Σ, \vec{z}, u) to M from a prestable holomorphic disc Σ with k boundary marked points $\vec{z} = (z_1, \dots, z_k)$, with $u(\partial\Sigma) \subseteq L$ and $[u(\Sigma)] = \beta$ in $H_2(M, L; \mathbb{Z})$. Using results of Fukaya, Oh, Ohta and Ono [10, §7–§8] involving their theory of Kuranishi spaces, we can make $\overline{\mathcal{M}}_k(M, L, J, \beta)$ into a compact d -orbifold with corners $\mathbf{\overline{\mathcal{M}}}_k(M, L, J, \beta)$. Given a relative spin structure for (M, L) , we may define an orientation on $\mathbf{\overline{\mathcal{M}}}_k(M, L, J, \beta)$.
- (iv) Let X be a complex projective manifold, and $\overline{\mathcal{M}}_{g,m}(X, \beta)$ the Deligne–Mumford moduli \mathbb{C} -stack of stable triples (Σ, \vec{z}, u) for Σ a genus g prestable Riemann surface with m marked points $\vec{z} = (z_1, \dots, z_m)$ and $u : \Sigma \rightarrow X$ a morphism with $u_*([\Sigma]) = \beta \in H_2(X; \mathbb{Z})$. Then Behrend [2] defines a perfect obstruction theory on $\overline{\mathcal{M}}_{g,m}(X, \beta)$, so we can make $\overline{\mathcal{M}}_{g,m}(X, \beta)$ into a compact, oriented d -orbifold $\mathbf{\overline{\mathcal{M}}}_{g,m}(X, \beta)$.
- (v) Let X be a complex algebraic surface, and \mathcal{M} a stable moduli \mathbb{C} -scheme of vector bundles or coherent sheaves E on X with fixed Chern character. Then Mochizuki [28] defines a perfect obstruction theory on \mathcal{M} , so we can make \mathcal{M} into an oriented d -manifold $\mathbf{\mathcal{M}}$.
- (vi) Let X be a complex Calabi–Yau 3-fold or smooth Fano 3-fold, and \mathcal{M} a stable moduli \mathbb{C} -scheme of coherent sheaves E on X with fixed Hilbert polynomial. Then Thomas [32] defines a perfect obstruction theory on \mathcal{M} , so we can make \mathcal{M} into an oriented d -manifold $\mathbf{\mathcal{M}}$.
- (vii) Let X be a smooth complex projective 3-fold, and \mathcal{M} a moduli \mathbb{C} -scheme of ‘stable PT pairs’ (C, D) in X , where $C \subset X$ is a curve and $D \subset C$ is a divisor. Then Pandharipande and Thomas [30] define a perfect obstruction theory on \mathcal{M} , so we can make \mathcal{M} into a compact, oriented d -manifold $\mathbf{\mathcal{M}}$.
- (ix) Let X be a complex Calabi–Yau 3-fold, and \mathcal{M} a separated moduli \mathbb{C} -scheme of simple perfect complexes in the derived category $D^b \text{coh}(X)$. Then Huybrechts and Thomas [16] define a perfect obstruction theory on \mathcal{M} , so we can make \mathcal{M} into an oriented d -manifold $\mathbf{\mathcal{M}}$.

We can use d -manifolds and d -orbifolds to construct *virtual classes* or *virtual chains* for all these moduli spaces.

Remark 4.45. D -manifolds should not be confused with *differential graded manifolds*, or *dg-manifolds*. This term is used in two senses, in algebraic geometry to mean a special kind of dg -scheme, as in Ciocan-Fontanine and Kapranov [7, Def. 2.5.1], and in differential geometry to mean a supermanifold with

extra structure, as in Cattaneo and Schätz [6, Def. 3.6]. In both cases, a dg-manifold \mathfrak{E} is roughly the total space of a graded vector bundle E^\bullet over a manifold V , with a vector field Q of degree 1 satisfying $[Q, Q] = 0$.

For example, if E is a vector bundle over V and $s \in C^\infty(E)$, we can make E into a dg-manifold \mathfrak{E} by giving E the grading -1 , and taking Q to be the vector field on E corresponding to s . To this \mathfrak{E} we can associate the d-manifold $S_{V,E,s}$ from Example 4.3. Note that $S_{V,E,s}$ only knows about an infinitesimal neighbourhood of $s^{-1}(0)$ in V , but \mathfrak{E} remembers all of V, E, s .

A Basics of 2-categories

Finally we discuss 2-categories. A good reference is Behrend et al. [3, App. B].

Definition A.1. A (*strict*) 2-category \mathfrak{C} consists of a proper class of *objects* $\text{Obj}(\mathfrak{C})$, for all $X, Y \in \text{Obj}(\mathfrak{C})$ a category $\text{Hom}(X, Y)$, for all X in $\text{Obj}(\mathfrak{C})$ an object id_X in $\text{Hom}(X, X)$ called the *identity 1-morphism*, and for all X, Y, Z in $\text{Obj}(\mathfrak{C})$ a functor $\mu_{X,Y,Z} : \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$. These must satisfy the *identity property*, that

$$\mu_{X,X,Y}(\text{id}_X, -) = \mu_{X,Y,Y}(-, \text{id}_Y) = \text{id}_{\text{Hom}(X,Y)}$$

as functors $\text{Hom}(X, Y) \rightarrow \text{Hom}(X, Y)$, and the *associativity property*, that

$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \text{id}_{\text{Hom}(Y,Z)}) = \mu_{W,X,Z} \circ (\text{id}_{\text{Hom}(W,X)} \times \mu_{X,Y,Z})$$

as functors $\text{Hom}(W, X) \times \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(W, Z)$.

Objects f of $\text{Hom}(X, Y)$ are called *1-morphisms*, written $f : X \rightarrow Y$. For 1-morphisms $f, g : X \rightarrow Y$, morphisms $\eta \in \text{Hom}_{\text{Hom}(X,Y)}(f, g)$ are called *2-morphisms*, written $\eta : f \Rightarrow g$. Thus, a 2-category has objects X , and two kinds of morphisms, 1-morphisms $f : X \rightarrow Y$ between objects, and 2-morphisms $\eta : f \Rightarrow g$ between 1-morphisms.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are 1-morphisms then $\mu_{X,Y,Z}(f, g)$ is the *horizontal composition of 1-morphisms*, written $g \circ f : X \rightarrow Z$. If $f, g, h : X \rightarrow Y$ are 1-morphisms and $\eta : f \Rightarrow g, \zeta : g \Rightarrow h$ are 2-morphisms then composition of η, ζ in $\text{Hom}(X, Y)$ gives the *vertical composition of 2-morphisms* of η, ζ , written $\zeta \odot \eta : f \Rightarrow h$, as a diagram

$$\begin{array}{ccc} \begin{array}{ccc} & f & \\ & \Downarrow \eta & \\ X & \xrightarrow{\quad g \quad} & Y \\ & \Downarrow \zeta & \\ & h & \end{array} & \rightsquigarrow & \begin{array}{ccc} & f & \\ & \Downarrow \zeta \odot \eta & \\ X & \xrightarrow{\quad \quad \quad} & Y \\ & h & \end{array} \end{array} \quad (\text{A.1})$$

And if $f, \tilde{f} : X \rightarrow Y$ and $g, \tilde{g} : Y \rightarrow Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde{f}, \zeta : g \Rightarrow \tilde{g}$ are 2-morphisms then $\mu_{X,Y,Z}(\eta, \zeta)$ is the *horizontal composition of*

2-morphisms, written $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$, as a diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \Downarrow \eta \\ \xrightarrow{\tilde{f}} \end{array} Y \begin{array}{c} \xrightarrow{g} \\ \Downarrow \zeta \\ \xrightarrow{\tilde{g}} \end{array} Z \quad \rightsquigarrow \quad X \begin{array}{c} \xrightarrow{g \circ f} \\ \Downarrow \zeta * \eta \\ \xrightarrow{\tilde{g} \circ \tilde{f}} \end{array} Z. \quad (\text{A.2})$$

There are also two kinds of identity: *identity 1-morphisms* $\text{id}_X : X \rightarrow X$ and *identity 2-morphisms* $\text{id}_f : f \Rightarrow f$.

A basic example is the 2-category of categories \mathfrak{Cat} , with objects categories \mathcal{C} , 1-morphisms functors $F : \mathcal{C} \rightarrow \mathcal{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$ for functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$. Orbifolds naturally form a 2-category, as do stacks in algebraic geometry.

In a 2-category \mathfrak{C} , there are three notions of when objects X, Y in \mathfrak{C} are ‘the same’: *equality* $X = Y$, and *isomorphism*, that is we have 1-morphisms $f : X \rightarrow Y$, $g : Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$, and *equivalence*, that is we have 1-morphisms $f : X \rightarrow Y$, $g : Y \rightarrow X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \text{id}_X$ and $\zeta : f \circ g \Rightarrow \text{id}_Y$. Usually equivalence is the correct notion.

Commutative diagrams in 2-categories should in general only commute *up to (specified) 2-isomorphisms*, rather than strictly. A simple example of a commutative diagram in a 2-category \mathfrak{C} is

$$\begin{array}{ccc} & & Y \\ & \nearrow f & \searrow g \\ X & & Z \\ & \searrow h & \nearrow \end{array} \quad \Downarrow \eta$$

which means that X, Y, Z are objects of \mathfrak{C} , $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : X \rightarrow Z$ are 1-morphisms in \mathfrak{C} , and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

We define fibre products in 2-categories, following [3, Def. B.13].

Definition A.2. Let \mathfrak{C} be a 2-category and $g : X \rightarrow Z$, $h : Y \rightarrow Z$ be 1-morphisms in \mathfrak{C} . A *fibre product* $X \times_Z Y$ in \mathfrak{C} consists of an object W , 1-morphisms $\pi_X : W \rightarrow X$ and $\pi_Y : W \rightarrow Y$ (we usually write $e = \pi_X$ and $f = \pi_Y$) and a 2-isomorphism $\eta : g \circ \pi_X \Rightarrow h \circ \pi_Y$ in \mathfrak{C} with the following universal property: suppose $\pi'_X : W' \rightarrow X$ and $\pi'_Y : W' \rightarrow Y$ are 1-morphisms and $\eta' : g \circ \pi'_X \Rightarrow h \circ \pi'_Y$ is a 2-isomorphism in \mathfrak{C} . Then there should exist a 1-morphism $b : W' \rightarrow W$ and 2-isomorphisms $\zeta_X : \pi_X \circ b \Rightarrow \pi'_X$, $\zeta_Y : \pi_Y \circ b \Rightarrow \pi'_Y$ such that the following diagram of 2-isomorphisms commutes:

$$\begin{array}{ccc} g \circ \pi_X \circ b & \xRightarrow{\eta * \text{id}_b} & h \circ \pi_Y \circ b \\ \text{id}_g * \zeta_X \Downarrow & & \Downarrow \text{id}_h * \zeta_Y \\ g \circ \pi'_X & \xRightarrow{\eta'} & h \circ \pi'_Y. \end{array}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}_X, \tilde{\zeta}_Y$ are alternative choices of b, ζ_X, ζ_Y then there should exist a unique 2-isomorphism $\theta : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta}_X = \zeta_X \odot (\text{id}_{\pi_X} * \theta) \quad \text{and} \quad \tilde{\zeta}_Y = \zeta_Y \odot (\text{id}_{\pi_Y} * \theta).$$

If a fibre product $X \times_Z Y$ in \mathfrak{C} exists then it is unique up to equivalence.

Orbifolds, and stacks in algebraic geometry, form 2-categories, and Definition A.2 is the right way to define fibre products of orbifolds or stacks.

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